

# ZETA FUNCTIONS AND OSCILLATORY INTEGRALS FOR MEROMORPHIC FUNCTIONS

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**ABSTRACT.** In the 70's Igusa developed a uniform theory for local zeta functions and oscillatory integrals attached to polynomials with coefficients in a local field of characteristic zero. In the present article this theory is extended to the case of rational functions, or, more generally, meromorphic functions  $f/g$ , with coefficients in a local field of characteristic zero. This generalization is far from being straightforward due to the fact that several new geometric phenomena appear. Also, the oscillatory integrals have two different asymptotic expansions: the ‘usual’ one when the norm of the parameter tends to infinity, and another one when the norm of the parameter tends to zero. The first asymptotic expansion is controlled by the poles (with negative real parts) of all the twisted local zeta functions associated to the meromorphic functions  $f/g - c$ , for certain special values  $c$ . The second expansion is controlled by the poles (with positive real parts) of all the twisted local zeta functions associated to  $f/g$ .

## 1. INTRODUCTION

In the present article we extend the theory of local zeta functions and oscillatory integrals to the case of *meromorphic* functions defined over local fields. In the classical setting, working on a local field  $K$  of characteristic zero, for instance on  $\mathbb{R}$ ,  $\mathbb{C}$  or  $\mathbb{Q}_p$ , one considers a pair  $(h, \Phi)$ , where  $h : U \rightarrow K$  is a  $K$ -analytic function defined on an open set  $U \subset K^n$  and  $\Phi$  is a test function, compactly supported in  $U$ . One attaches to  $(h, \Phi)$  the local zeta function

$$Z_\Phi(s; h) := \int_{K^n \setminus h^{-1}\{0\}} \Phi(x) |h(x)|_K^s |dx|_K$$

for  $s \in \mathbb{C}$  with  $\operatorname{Re}(s) > 0$ . Here  $|dx|_K$  denotes the Haar measure on  $K^n$  and  $|a|_K$  the modulus of  $a \in K$ . More generally, one considers the (twisted) local zeta function

$$Z_\Phi(\omega; h) := \int_{K^n \setminus h^{-1}\{0\}} \Phi(x) \omega(h(x)) |dx|_K$$

for a quasi-character  $\omega$  of  $K^\times$ . By using an embedded resolution of singularities of  $h^{-1}\{0\}$ , one shows that  $Z_\Phi(\omega; h)$  admits a meromorphic continuation to the whole

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complex plane. In the Archimedean case the poles of  $Z_\Phi(\omega; h)$  are integer shifts of the roots of the Bernstein-Sato polynomial of  $h$ , and hence induce eigenvalues of the complex monodromy of  $h$ . In the  $p$ -adic case such a connection has been established in several cases, but a general proof constitutes one of the major challenges in this area.

In the Archimedean case  $K = \mathbb{R}$  or  $\mathbb{C}$ , the study of local zeta functions was initiated by I. M. Gel'fand and G. E. Shilov [23]. The meromorphic continuation of the local zeta functions was established, independently, by M. Atiyah [3] and J. Bernstein [6], see also [31, Theorem 5.5.1 and Corollary 5.5.1]. On the other hand, in the middle 60's, A. Weil initiated the study of local zeta functions, in the Archimedean and non-Archimedean settings, in connection with the Poisson-Siegel formula [43]. In the 70's, J.-I. Igusa developed (for polynomials  $h$ ) a uniform theory for local zeta functions over local fields of characteristic zero [29], [31]. The  $p$ -adic local zeta functions are connected with the number of solutions of polynomial congruences mod  $p^m$  and with exponential sums mod  $p^m$ , see e.g. [14], [31]. More recently, J. Denef and F. Loeser introduced in [19] the motivic zeta functions which constitute a vast generalization of the  $p$ -adic local zeta functions.

Fixing an additive character  $\Psi : K \rightarrow \mathbb{C}$ , the oscillatory integral attached to  $(h, \Phi)$  is defined as

$$E_\Phi(z; h) = \int_{K^n \setminus h^{-1}\{0\}} \Phi(x) \Psi(zh(x)) |dx|_K \quad \text{for } z \in K.$$

A central mathematical problem consists in describing the asymptotic behavior of  $E_\Phi(z; h)$  as  $|z|_K \rightarrow \infty$ . Under mild conditions, there exists an asymptotic expansion of  $E_\Phi(z; h)$  at infinity, controlled by the poles of twisted local zeta functions. For instance when  $K = \mathbb{Q}_p$  we have the following [29]. Assume that (the restriction to the support of  $\Phi$  of) the critical locus of  $h$  is contained in  $h^{-1}\{0\}$ , and denote for simplicity  $|z| = |z|_{\mathbb{Q}_p}$ . Then

$$(1.1) \quad E_\Phi(z; h) = \sum_{\gamma} \sum_{m=1}^{m_\gamma} e_{\gamma, m} \left( \frac{z}{|z|} \right) |z|^\gamma (\ln |z|)^{m-1}$$

for sufficiently large  $|z|$ , where  $\gamma$  runs through all the poles mod  $2\pi i / \ln p$  of  $Z_\Phi(\omega; h)$  (for all quasi-characters  $\omega$ ),  $m_\gamma$  is the order of  $\gamma$ , and  $e_{\gamma, m}$  is a locally constant function on  $\mathbb{Z}_p^\times$ .

In this article we extend Igusa's theory for local zeta functions and oscillatory integrals to the case in which the function  $h$  is replaced by a meromorphic function. (Actually, Igusa's theory is developed in [29], [31] for polynomials  $h$ , but it is more generally valid for analytic functions, since the arguments are locally analytic on a resolution space.) Besides independent interest, there are other mathematical and physical motivations for pursuing this line of research. S. Gusein-Zade, I. Luengo and A. Melle-Hernández have studied the complex monodromy (and A'Campo zeta functions attached to it) of meromorphic functions, see e.g. [25], [26], [27]. This work drives naturally to ask about the existence of local zeta functions with poles related with the monodromies studied by the mentioned authors. At an arithmetic level, we mention the special case of the oscillatory integrals associated to  $p$ -adic Laurent polynomials, that are in fact exponential sums mod  $p^\ell$ . Estimates for exponential sums attached to non-degenerate Laurent polynomials mod  $p$  were obtained

by A. Adolphson and S. Sperber [1] and J. Denef and F. Loeser [17]. Finally, the local zeta functions attached to meromorphic functions are very alike to parametric Feynman integrals and to  $p$ -adic string amplitudes, see e.g. [5], [7], [9], [37]. For instance in [37, Section 3.15], M. Marcolli pointed out explicitly that the motivic Igusa zeta function constructed by J. Denef and F. Loeser may provide the right tool for a motivic formulation of the dimensionally regularized parametric Feynman integrals.

We now describe our results. In Section 3, we describe the meromorphic continuation of (twisted) local zeta functions attached to meromorphic functions  $f/g$ , see Theorems 3.2 and 3.5. These results are related with the work of F. Loeser in [36] for multivariable local zeta functions. The local zeta functions attached to meromorphic functions may have poles with positive and negative real parts. We establish the existence of a band  $\beta < \operatorname{Re}(s) < \alpha$ , with  $\alpha \in \mathbb{Q}_{>0} \cup \{+\infty\}$  and  $\beta \in \mathbb{Q}_{<0} \cup \{-\infty\}$ , on which the local zeta functions are analytic, and we show that  $\alpha, \beta$  can be read off from an embedded resolution of singularities of the divisor defined by  $f$  and  $g$ , see Theorem 3.9. In the case of meromorphic functions, the problem of determining whether or not the corresponding local zeta function has a pole is more complicated than in the function case, see Example 3.13. We develop criteria for the existence of poles, see Corollary 3.12 and Lemma 3.14. In Subsection 3.5, we treat briefly the motivic and topological zeta functions for rational functions. These invariants are connected with special cases of the general theory of motivic integration of R. Cluckers and F. Loeser [12].

Sections 4 and 5 constitute the core of the article. In Section 4, we review Igusa's method for estimating oscillatory integrals for polynomials/holomorphic functions, and we present our new strategy and several technical results in the case of meromorphic functions, see Propositions 4.9, 4.10, 4.12 and 4.13. Then in Section 5, we prove our expansions and estimations for oscillatory integrals associated to meromorphic functions. This is not a straightforward matter, due to the fact that new geometric phenomena occur in the meromorphic case, see Remarks 3.1, 4.7 and 5.6, and Definitions 5.7 and 5.10. In addition, there exist two different asymptotic expansions: one when the parameter of the oscillatory integral approaches the origin and another when this parameter approaches infinity. The first asymptotic expansion is controlled by the poles with positive real parts of all twisted local zeta functions attached to the corresponding meromorphic function, see Theorems 5.2, 5.3 and 5.17. The second expansion is controlled by the poles with negative real parts of all twisted local zeta functions attached to the corresponding meromorphic function, see Theorems 5.11, 5.12 and 5.18. As an illustration, we mention here the form of the expansion at infinity when  $K = \mathbb{Q}_p$ . Let now  $h$  be a meromorphic function on  $U$ . In Definition 5.10 we associate a finite set  $\mathcal{V}$  of *special values* to  $h$  and  $\Phi$ , including the critical values of (a resolution of indeterminacies of)  $h$  that belong to the support of  $\Phi$ . Then

$$(1.2) \quad E_{\Phi}(z; h) = \sum_{c \in \mathcal{V}} \sum_{\gamma_c < 0} \sum_{m=1}^{m_{\gamma_c}} e_{\gamma_c, m, c} \left( \frac{z}{|z|} \right) \Psi(c \cdot z) |z|^{\gamma_c} (\ln |z|)^{m-1}$$

for sufficiently large  $|z|$ , where  $\gamma_c$  runs through all the poles mod  $2\pi\sqrt{-1}/\ln p$  with negative real part of  $Z_{\Phi}(\omega; h - c)$  (for all quasi-characters  $\omega$ ),  $m_{\gamma_c}$  is the order of  $\gamma_c$ , and  $e_{\gamma_c, m, c}$  is a locally constant function on  $\mathbb{Z}_p^{\times}$ . Consider for example

$h = (x^2 + x^3 - y^2)/x^2$ . Then 1 is the only special value (it is not a critical value), and Theorem 5.12 yields the term  $\Psi(z)|z|^{-5/2}$  in the expansion, see Examples 5.8 and 5.15.

Note that the expansion (1.1) is simpler, mainly due to restricting the support of  $\Phi$ . In the context of meromorphic functions however, one typically has expansions as in (1.2), even when the support of  $\Phi$  is arbitrarily small!

In [13], see also [11], R. Cluckers and F. Loeser obtained similar expansions to those presented in Theorem 5.12, for  $p$  big enough, in a more general non-Archimedean setting, but without information on the powers of  $|z|_K$  and  $\ln|z|_K$ . We also note that the existence of two types of asymptotic expansions for  $p$ -adic oscillatory integrals attached to Laurent polynomials, satisfying certain very restrictive conditions, was established by E. León-Cardenal and the second author in [34].

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## 2. PRELIMINARIES

**2.1. The group of quasicharacters of a local field.** We take  $K$  to be a non-discrete locally compact field of characteristic zero. Then  $K$  is  $\mathbb{R}$ ,  $\mathbb{C}$ , or a finite extension of  $\mathbb{Q}_p$ , the field of  $p$ -adic numbers. If  $K$  is  $\mathbb{R}$  or  $\mathbb{C}$ , we say that  $K$  is an  $\mathbb{R}$ -field, otherwise we say that  $K$  is a  $p$ -field.

For  $a \in K$ , we define the *modulus*  $|a|_K$  of  $a$  by

$$|a|_K = \begin{cases} \text{the rate of change of the Haar measure in } (K, +) \text{ under } x \rightarrow ax \\ \text{for } a \neq 0, \\ 0 \text{ for } a = 0. \end{cases}$$

It is well-known that, if  $K$  is an  $\mathbb{R}$ -field, then  $|a|_{\mathbb{R}} = |a|$  and  $|a|_{\mathbb{C}} = |a|^2$ , where  $|\cdot|$  denotes the usual absolute value in  $\mathbb{R}$  or  $\mathbb{C}$ , and, if  $K$  is a  $p$ -field, then  $|\cdot|_K$  is the normalized absolute value in  $K$ .

A quasicharacter of  $K^\times$  is a continuous group homomorphism from  $K^\times$  into  $\mathbb{C}^\times$ . The set of quasicharacters forms a topological Abelian group denoted by  $\Omega(K^\times)$ . The connected component of the identity consists of the  $\omega_s$ ,  $s \in \mathbb{C}$ , defined by  $\omega_s(z) = |z|_K^s$  for  $z \in K^\times$ .

We now take  $K$  to be a  $p$ -field. Let  $R_K$  be the valuation ring of  $K$ ,  $P_K$  the maximal ideal of  $R_K$ , and  $\overline{K} = R_K/P_K$  the residue field of  $K$ . The cardinality of the residue field of  $K$  is denoted by  $q$ , thus  $\overline{K} = \mathbb{F}_q$ . For  $z \in K$ ,  $\text{ord}(z) \in \mathbb{Z} \cup \{+\infty\}$  denotes the valuation of  $z$ , and  $|z|_K = q^{-\text{ord}(z)}$ . We fix a uniformizing parameter  $\mathfrak{p}$  of  $R_K$ . If  $z \in K^\times$ , then  $ac\,z = z\mathfrak{p}^{-\text{ord}(z)}$  denotes the angular component of  $z$ .

Given  $\omega \in \Omega(K^\times)$ , we choose  $s \in \mathbb{C}$  satisfying  $\omega(\mathfrak{p}) = q^{-s}$ . Then  $\omega$  can be described as follows:  $\omega(z) = \omega_s(z)\chi(ac\,z)$  for  $z \in K^\times$ , in which  $\chi := \omega|_{R_K^\times}$  is a character of  $R_K^\times$ . Furthermore,  $\Omega(K^\times)$  is a one dimensional complex manifold, isomorphic to  $(\mathbb{C} \bmod (2\pi i/\ln q)\mathbb{Z}) \times (R_K^\times)^*$ , where  $(R_K^\times)^*$  is the group of characters of  $R_K^\times$ .

Next we take  $K$  to be an  $\mathbb{R}$ -field. Now for  $z \in K^\times$  we denote  $ac\,z = \frac{z}{|z|}$ . Then  $\omega \in \Omega(K^\times)$  can again be described as  $\omega(z) = \omega_s(z)\chi(ac\,z)$  for  $z \in K^\times$ , where

$\chi$  is a character of  $\{z \in K^\times \mid |z| = 1\}$ . Concretely in this case  $\chi = \chi_l = (\cdot)^l$ , in which  $l \in \{0, 1\}$  or  $l \in \mathbb{Z}$ , according as  $K = \mathbb{R}$  or  $K = \mathbb{C}$ . In addition,  $\Omega(K^\times)$  is a one dimensional complex manifold, which is isomorphic to  $\mathbb{C} \times (\mathbb{Z}/2\mathbb{Z})$  or  $\mathbb{C} \times \mathbb{Z}$ , according as  $K$  is  $\mathbb{R}$  or  $\mathbb{C}$ .

For arbitrary  $K$ , we will denote the decompositions above as  $\omega = \omega_s \chi(ac) \in \Omega(K^\times)$ . We have that  $\sigma(\omega) := \text{Re}(s)$  depends only on  $\omega$ , and  $|\omega(z)| = \omega_{\sigma(\omega)}(z)$ . We define for all  $\beta < \alpha$  in  $\mathbb{R} \cup \{-\infty, +\infty\}$  an open subset of  $\Omega(K^\times)$  by

$$\Omega_{(\beta, \alpha)}(K^\times) = \{\omega \in \Omega(K^\times) \mid \sigma(\omega) \in (\beta, \alpha)\}.$$

For further details we refer the reader to [31].

**2.2. Local zeta functions for meromorphic functions.** If  $K$  is a  $p$ -field, resp. an  $\mathbb{R}$ -field, we denote by  $\mathcal{D}(K^n)$  the  $\mathbb{C}$ -vector space consisting of all  $\mathbb{C}$ -valued locally constant functions, resp. all smooth functions, on  $K^n$ , with compact support. An element of  $\mathcal{D}(K^n)$  is called a *test function*. To simplify terminology, we will call a non-zero test function that takes only real and non-negative values a *positive* test function.

Let  $f, g : U \rightarrow K$  be non-zero  $K$ -analytic functions defined on an open  $U$  in  $K^n$ , such that  $f/g$  is not constant. Let  $\Phi : U \rightarrow \mathbb{C}$  be a test function with support in  $U$ . Then the local zeta function attached to  $(\omega, f/g, \Phi)$  is defined as

$$(2.1) \quad Z_\Phi(\omega; f/g) = \int_{U \setminus D_K} \Phi(x) \omega\left(\frac{f(x)}{g(x)}\right) |dx|_K,$$

where  $D_K = f^{-1}\{0\} \cup g^{-1}\{0\}$  and  $|dx|_K$  is the normalized Haar measure on  $K^n$ .

**Remark 2.1.** (1) The convergence of the integral in (2.1) is not a straightforward matter; in particular the convergence does not follow from the fact that  $\Phi$  has compact support.

(2) When considering only polynomials  $f$  and  $g$ , it would be natural to assume that  $f$  and  $g$  are coprime in the polynomial ring  $K[x_1, \dots, x_n]$ . In that case the set  $D_K$  only depends on  $f/g$ . For more general  $K$ -analytic functions however, the set  $D_K$  depends in fact on the chosen  $f$  and  $g$  to represent the meromorphic function  $f/g$ . But this will not really affect our methods and results. Note that the zeta function  $Z_\Phi(\omega; f/g)$  does depend only on the quotient  $f/g$ .

**2.3. Ordinary and adapted embedded resolutions.** We state two versions of embedded resolution of  $D_K$  that we will use in this paper.

**Theorem 2.2.** *Let  $U$  be an open subset of  $K^n$ . Let  $f, g$  be  $K$ -analytic functions on  $U$  as in Subsection 2.2.*

- (1) *Then there exists an embedded resolution  $\sigma : X_K \rightarrow U$  of  $D_K$ , that is,*
  - (i)  *$X_K$  is an  $n$ -dimensional  $K$ -analytic manifold,  $\sigma$  is a proper  $K$ -analytic map which is locally a composition of a finite number of blow-ups at closed submanifolds, and which is an isomorphism outside of  $\sigma^{-1}(D_K)$ ;*
  - (ii)  *$\sigma^{-1}(D_K) = \cup_{i \in T} E_i$ , where the  $E_i$  are closed submanifolds of  $X_K$  of codimension one, each equipped with a pair of nonnegative integers  $(N_{f,i}, N_{g,i})$  and a positive integer  $v_i$ , satisfying the following. At every point  $b$  of  $X_K$  there exist local coordinates  $(y_1, \dots, y_n)$  on  $X_K$  around  $b$  such that, if  $E_1, \dots, E_r$  are the  $E_i$  containing  $b$ , we have on some open neighborhood  $V$  of  $b$  that  $E_i$  is given by  $y_i = 0$  for*

$i \in \{1, \dots, r\}$ ,

$$(2.2) \quad \sigma^*(dx_1 \wedge \dots \wedge dx_n) = \eta(y) \left( \prod_{i=1}^r y_i^{v_i-1} \right) dy_1 \wedge \dots \wedge dy_n,$$

and

$$(2.3) \quad f^*(y) := (f \circ \sigma)(y) = \varepsilon_f(y) \prod_{i=1}^r y_i^{N_{f,i}},$$

$$(2.4) \quad g^*(y) := (g \circ \sigma)(y) = \varepsilon_g(y) \prod_{i=1}^r y_i^{N_{g,i}},$$

where  $\eta(y), \varepsilon_f(y), \varepsilon_g(y)$  belong to  $\mathcal{O}_{X_K, b}^\times$ , the group of units of the local ring of  $X_K$  at  $b$ .

(2) Furthermore, we can construct such an embedded resolution  $\sigma : X_K \rightarrow U$  of  $D_K$  satisfying the following additional property at every point  $b$  of  $X_K$ :

(iii) with the notation of (2.3) and (2.4), either  $f^*(y)$  divides  $g^*(y)$  in  $\mathcal{O}_{X_K, b}$  (what is equivalent to  $N_{f,i} \leq N_{g,i}$  for all  $i = 1, \dots, r$ ), or conversely  $g^*(y)$  divides  $f^*(y)$ .

*Proof.* Part (1) is one of the standard formulations of embedded resolution. It follows from Hironaka's work [28]. See also [8, Section 8], [22, Section 5], [45], [46] for more details, and especially [16, Theorem 2.2] for the  $p$ -field case.

One obtains a resolution as in part (2) by first resolving the indeterminacies of  $f/g$ , considered as map from  $U$  to the projective line, and then further computing an embedded resolution of the union of the exceptional locus and the strict transform of  $D_K$ .  $\square$

**Remark 2.3.** (1) When  $f$  and  $g$  are polynomials, the map  $\sigma$  is a composition of a finite number of blow-ups. But in the more general analytic setting, it is possible that one needs infinitely many blow-ups, and hence that  $T$  is infinite. Consider for example the case  $U = \{(a, b) \in \mathbb{R}^2 \mid a \neq 0\}$ ,  $f = (y-1)(y - \sin 1/x)$  and  $g = 1$ . The curve defined by  $f$  has infinitely many isolated singular points, even contained in a bounded part of  $U$ .

There are even such examples where the multiplicity of the singular points is not bounded. Consider the analytic function  $a(x) = \prod_{j=1}^\infty jx \sin \frac{1}{jx}$  on  $\mathbb{R}_{>0}$ , with zeroes at  $\frac{1}{m\pi}$ ,  $m \in \mathbb{Z}_{>0}$ . For instance  $\frac{1}{m\pi}$  is a zero of multiplicity  $m$ . Then the curve defined by  $a(x) - a(y)$  on  $U = \mathbb{R}_{>0}^2$  has infinitely many singular points contained in a bounded part of  $U$ , where moreover the multiplicities of those singular points are not bounded.

(2) However, the construction of  $\sigma$  is locally finite in the following sense. For any compact set  $\mathcal{C} \subset U$ , there is an open neighborhood  $U_{\mathcal{C}} \supset \mathcal{C}$  such that the restriction of  $\sigma$  to  $\sigma^{-1}(U_{\mathcal{C}})$  is a composition of a finite number of blow-ups. We refer to [46] for more details. In order to handle this situation we note that in our setting

(i) the objects of study associated to  $f$  and  $g$ , zeta functions and oscillatory integrals, depend only on the values of  $f$  and  $g$  on a small neighborhood of the (compact) support of the test function  $\Phi$ ; and

(ii) most of our invariants and estimates depend on  $\Phi$ .

CONVENTION. We consider here simply the compact set  $\mathcal{C} := \text{supp } \Phi$  and we fix an appropriate open  $U_{\Phi} := U_{\mathcal{C}}$  such that the restriction of  $\sigma$  to  $\sigma^{-1}(U_{\Phi})$  is a composition of a finite number of blow-ups.

**Definition 2.4.** In the sequel we will call an embedded resolution  $\sigma$  as in part (2) of Theorem 2.2 an *adapted embedded resolution* of  $D_K$ .

**Remark 2.5.** An adapted embedded resolution yields in general much more components  $E_i, i \in T$ , than an ‘economic’ standard embedded resolution. But it will be a crucial tool to derive the results of Sections 4 and 5.

**Remark 2.6.** In Theorem 2.2, we note that  $f^*/g^*$ , considered as a map to  $K$ , is defined in the points  $b$  of  $\sigma^{-1}(D_K)$  satisfying  $N_{f,i} \geq N_{g,i}$  for  $i = 1, \dots, r$ . In particular there can exist points  $b \in \sigma^{-1}(D_K)$  such that  $\frac{f^*(y)}{g^*(y)}$  is a unit in the local ring at  $b$  (this happens when  $N_{f,i} = N_{g,i}$  for  $i = 1, \dots, r$ ). Moreover, in that case the degree of the first non-constant term in the Taylor expansion at  $b$  can be larger than 1. See Example 5.14.

This will be an important new feature when studying zeta functions and oscillatory integrals of meromorphic functions  $f/g$ , compared to analytic functions  $f$ .

**Definition 2.7.** Let  $\sigma : X_K \rightarrow U$  be an embedded resolution of  $D_K$  as in Theorem 2.2 and Remark 2.3. For  $i \in T$ , we denote  $N_i = N_{f,i} - N_{g,i}$  and call  $(N_i, v_i)$  the datum of  $E_i$ .

(1) We put  $T_+ = \{i \in T \mid N_i > 0 \text{ and } E_i \cap \sigma^{-1}(\text{supp } \Phi) \neq \emptyset\}$  and  $T_- = \{i \in T \mid N_i < 0 \text{ and } E_i \cap \sigma^{-1}(\text{supp } \Phi) \neq \emptyset\}$ , and we define

$$\alpha = \alpha_\Phi(\sigma, D_K) = \begin{cases} \min_{i \in T_-} \left\{ \frac{v_i}{|N_i|} \right\} & \text{if } T_- \neq \emptyset \\ +\infty & \text{if } T_- = \emptyset, \end{cases}$$

and

$$\beta = \beta_\Phi(\sigma, D_K) = \begin{cases} \max_{i \in T_+} \left\{ -\frac{v_i}{N_i} \right\} & \text{if } T_+ \neq \emptyset \\ -\infty & \text{if } T_+ = \emptyset. \end{cases}$$

(2) Whenever  $T$  is finite, in particular when  $f$  and  $g$  are polynomials, we can remove the condition  $E_i \cap \sigma^{-1}(\text{supp } \Phi) \neq \emptyset$  in the definition of  $T_+$  and  $T_-$ , and then we obtain global invariants  $\alpha$  and  $\beta$  not depending on  $\Phi$ .

### 3. CONVERGENCE, MEROMORPHIC CONTINUATION AND POLES OF LOCAL ZETA FUNCTIONS

**3.1. Zeta functions over  $p$ -fields.** Before treating general zeta functions for meromorphic functions, it is useful to recall the following basic computation.

**Lemma 3.1** (Lemma 8.2.1 in [31]). *Let  $K$  be a  $p$ -field. Take  $a \in K$ ,  $\omega = \omega_s \chi(ac) \in \Omega(K^\times)$  and  $N \in \mathbb{Z}$ . Take also  $n \in \mathbb{Z}_{>0}$  and  $e \in \mathbb{Z}_{\geq 0}$ . Then*

$$\int_{(a + \mathfrak{p}^e R_K) \setminus \{0\}} \omega(z)^N |z|_K^{n-1} |dz| = \begin{cases} (1 - q^{-1}) \frac{q^{-en - eNs}}{1 - q^{-n - Ns}} & \text{if } \begin{matrix} a \in \mathfrak{p}^e R_K \\ \chi^N = 1 \end{matrix} \\ q^{-e} \omega(a)^N |a|_K^{n-1} & \text{if } \begin{matrix} a \notin \mathfrak{p}^e R_K \\ \chi^N|_{U'} = 1 \end{matrix} \\ 0 & \text{all other cases,} \end{cases}$$

in which  $U' = 1 + \mathfrak{p}^e a^{-1} R_K$ . In the first case, the integral converges on  $\operatorname{Re}(s) > -\frac{n}{N}$ , if  $N > 0$ , and on  $\operatorname{Re}(s) < \frac{n}{|N|}$ , if  $N < 0$ . Note that for  $N = 0$  we obtain a non-zero constant.

**Theorem 3.2.** *Assume that  $K$  is a  $p$ -field. We consider  $f, g$  as in Section 2, and we fix an embedded resolution  $\sigma$  for  $D_K$  as in Theorem 2.2, for which we also use the notation of Definition 2.7. Then the following assertions hold:*

- (1)  $Z_\Phi(\omega; f/g)$  converges for  $\omega \in \Omega_{(\beta, \alpha)}(K^\times)$ ;
- (2)  $Z_\Phi(\omega; f/g)$  has a meromorphic continuation to  $\Omega(K^\times)$  as a rational function of  $\omega(\mathfrak{p}) = q^{-s}$ , and its poles are of the form

$$s = -\frac{v_i}{N_i} + \frac{2\pi\sqrt{-1}}{N_i \ln q} k, \quad k \in \mathbb{Z},$$

for  $i \in T_+ \cup T_-$ . In addition, the order of any pole is at most  $n$ .

*Proof.* These are more or less immediate consequences of the work of Loeser on multivariable zeta functions [36]. In his setting  $f$  and  $g$  are polynomials, but his arguments are also valid for holomorphic functions. Take  $\omega_1, \omega_2 \in \Omega(K^\times)$ . Then, by [36, Théorème 1.1.4], the integral

$$(3.1) \quad Z_\Phi(\omega_1, \omega_2; f, g) := \int_{U \setminus D_K} \Phi(x) \omega_1(f(x)) \omega_2(g(x)) |dx|_K,$$

obviously converging when  $\sigma(\omega_1)$  and  $\sigma(\omega_2)$  are positive, has a meromorphic continuation to  $\Omega(K^\times) \times \Omega(K^\times)$  as a rational function of  $\omega_1(\mathfrak{p}) = q^{-s_1}$  and  $\omega_2(\mathfrak{p}) = q^{-s_2}$ , with more precisely

$$\prod_i (1 - q^{v_i + N_{f,i}s_1 + N_{g,i}s_2})$$

as denominator; here  $i$  runs over the  $i \in T$  such that  $E_i \cap \sigma^{-1}(\operatorname{supp} \Phi) \neq \emptyset$ . Hence the real parts of the poles of  $Z_\Phi(\omega_1, \omega_2; f, g)$  belong to the union of the lines  $v_i + N_{f,i}s_1 + N_{g,i}s_2 = 0$ . Taking  $\omega_1 = \omega$  and  $\omega_2 = \omega^{-1}$  (and hence  $s_1 = s$  and  $s_2 = -s$ ), this specializes to the stated results about  $Z_\Phi(\omega; f/g)$ .  $\square$

**Remark 3.3.** Alternatively, it is straightforward to adapt Igusa's proof in the classical case ( $g = 1$ ) directly to our situation. As a preparation for Sections 4 and 5, we recall briefly the main idea.

(1) We pull back the integral  $Z_\Phi(\omega; f/g)$  to an integral over  $X_K \setminus \sigma^{-1}(D_K)$  via the resolution  $\sigma$ , and compute it by subdividing the (compact) integration domain  $\sigma^{-1}(\operatorname{supp} \Phi) \subset X_K$  in a finite disjoint union of compact open sets on which the integrand becomes 'monomial' in local coordinates. More precisely, using the notation of Theorem 2.2, we can assume that such an integration domain around a point  $b \in X_K$  is of the form  $B = c + (\mathfrak{p}^e R_K)^n$  in the local coordinates  $y_1, \dots, y_n$ , and that  $|\eta(y)|_K, |\varepsilon_f(y)|_K, |\varepsilon_g(y)|_K$  are constant on  $B$ . Then the contribution of  $B$  to  $Z_\Phi(\omega; f/g)$  is a non-zero constant times

$$(3.2) \quad \prod_{i=1}^r \int_{c_i + \mathfrak{p}^e R_K \setminus \{y_i=0\}} \omega^{N_i}(y_i) |y_i|_K^{v_i-1} |dy_i|_K.$$

Finally one concludes by using the local computation of Lemma 3.1.

We want to stress the new feature mentioned in Remark 2.6: in (3.2), it is possible that all  $N_i = 0$ , while some  $v_i > 1$ .



(2) Note that one needs an argument as in (1) to see that the defining integral of  $Z_\Phi(\omega; f/g)$  converges at least somewhere.

(3) If  $\sigma$  is an adapted embedded resolution, we have around  $b \in \sigma^{-1}(D_K)$  that  $N_1, \dots, N_r$  are either all non-positive or all non-negative in (3.2).

**Remark 3.4.** There is a refinement concerning the list of candidate poles in Theorem 3.2 and Remark 3.3. Writing  $\omega = \omega_s \chi(ac)$ , we have, by Lemma 3.1, that  $-\frac{v_i}{N_i}$  can be the real part of a pole of the corresponding integral in (3.2) only if the order of  $\chi$  divides  $N_i$ . Hence, in Theorem 3.2(2) the poles are subject to the additional restriction that the order of  $\chi$  divides  $N_i$ .

For later use in Sections 4 and 5, we stress the following special case. When  $N_i = 1$  in (3.2), the corresponding integral has no pole unless  $\chi$  is trivial. Then, considering in Theorem 3.2 the case that  $g = 1$  and  $f^{-1}\{0\} \cap \text{supp } \Phi$  has no singular points, we have that  $Z_\Phi(\omega_s \chi(ac); f)$  has no poles unless  $\chi$  is trivial, in which case its poles are of the form  $s = -1 + \frac{2\pi\sqrt{-1}}{\ln q}k$ ,  $k \in \mathbb{Z}$ , and of order 1.

**3.2. Zeta functions over  $\mathbb{R}$ -fields.** The strategy being analogous as for  $p$ -fields, we provide less details.

**Theorem 3.5.** *Assume that  $K$  is an  $\mathbb{R}$ -field. We consider  $f, g$  as in Section 2, and we fix an embedded resolution  $\sigma$  for  $D_K$  as in Theorem 2.2, for which we also use the notation of Definition 2.7. Then the following assertions hold:*

- (1)  $Z_\Phi(\omega; f/g)$  converges for  $\omega \in \Omega_{(\beta, \alpha)}(K^\times)$ ;
- (2)  $Z_\Phi(\omega, f/g)$  has a meromorphic continuation to  $\Omega(K^\times)$ , and its poles are of the form

$$s = -\frac{v_i}{N_i} - \frac{k}{[K : \mathbb{R}]N_i}, \quad k \in \mathbb{Z}_{\geq 0},$$

for  $i \in T_+ \cup T_-$ . In addition, the order of any pole is at most  $n$ .

*Proof.* Analogously as in the proof of Theorem 3.2, this can be derived from the results on multivariable zeta functions [36], although there the detailed form of the possible poles is not mentioned explicitly. Anyway, (2) can be shown exactly as in the proof of [31, Theorem 5.4.1].  $\square$

**Remark 3.6.** Again, as a preparation for Sections 4 and 5, we mention the main idea of (the generalization of) Igusa's proof in the classical case.

After pulling back the integral  $Z_\Phi(\omega; f/g)$  to an integral over  $X_K \setminus \sigma^{-1}(D_K)$  via the resolution  $\sigma$ , subdividing the new integration domain, and this time also using a partition of the unity, one writes  $Z_\Phi(\omega; f/g)$  as a finite linear combination of 'monomial' integrals. With the notation of Theorem 2.2, we can assume that these are of the form

$$\int_{K^n \setminus \cup_{i=1}^r \{y_i=0\}} \Theta(y) \omega(\epsilon_f(y) \epsilon_g^{-1}(y)) \prod_{i=1}^r \omega^{N_i}(y_i) \prod_{i=1}^r |y_i|_K^{v_i-1} |dy|_K,$$

where  $\Theta(y)$  is a smooth function with support in the polydisc

$$\{y \in K^n \mid |y_j|_K < 1 \text{ for } j = 1, \dots, n\}.$$

Note again that, following Remark 2.6, it is possible that all  $N_i = 0$ , while some  $v_i > 1$ .

**Remark 3.7.** Looking more in detail at the proof of [31, Theorem 5.4.1], which uses Bernstein polynomial techniques for dealing with the poles of monomial integrals, we have in fact a somewhat sharper result concerning the poles of  $Z_\Phi(\omega; f/g)$  when  $K = \mathbb{C}$ : when  $\omega(z) = |z|_K^s \left(\frac{z}{|z|}\right)^l$ , they are of the form  $s = -\frac{|l|}{2} - \frac{v_i+k}{N_i}, k \in \mathbb{Z}_{\geq 0}$ , for  $N_i > 0$ , and  $s = \frac{|l|}{2} + \frac{v_i+k}{|N_i|}, k \in \mathbb{Z}_{\geq 0}$ , for  $N_i < 0$ .

**Remark 3.8.** For later use in Sections 4 and 5, we consider in Theorem 3.5 the case that  $g = 1$  and  $f^{-1}\{0\} \cap \text{supp } \Phi$  has no singular points. If  $K = \mathbb{C}$ , we have as special case of Remark 3.7 that the poles of  $Z_\Phi(\omega_s \chi_l(ac); f)$  are of the form  $s = -\frac{|l|}{2} - 1 - k, k \in \mathbb{Z}_{\geq 0}$ . If  $K = \mathbb{R}$ , the poles of  $Z_\Phi(\omega_s \chi_l(ac); f)$  are odd integers when  $l = 0$  and even integers when  $l = 1$ . Both when  $K = \mathbb{C}$  and  $K = \mathbb{R}$  these poles are of order 1 [32, Theorem 4.19], [2, II §7].

Here we also want to mention that there is substantial work of Barlet and his collaborators related to Theorem 3.5 and the remarks above, see e.g. [4].

**3.3. Existence of poles, largest and smallest poles.** Take  $f, g : U \rightarrow K$  and a non-zero test function  $\Phi$  as in Section 2. We consider

$$Z_\Phi(s; f/g) := Z_\Phi(\omega_s; f/g) = \int_{U \setminus D_K} \Phi(x) \left| \frac{f(x)}{g(x)} \right|_K^s |dx|_K$$

for  $s \in \mathbb{C}$ . By the results of the previous subsections, we know that this integral converges when  $\beta < \text{Re}(s) < \alpha$ , where  $\alpha$  and  $\beta$  are as in Definition 2.7, hence a priori depending on some chosen embedded resolution.

It turns out that  $\alpha$  and  $\beta$  in fact *do not* depend on the chosen resolution. This follows from the next result, that generalizes the classical result for the zeta function of an analytic function  $f$ . For that classical result Igusa gives in [30] the strategy of a proof using Langlands' description of residues in terms of principal value integrals [33], see also [2] for  $K = \mathbb{R}$ . This idea uses explicit meromorphic continuations of certain monomial integrals, and it could probably be extended to the case of meromorphic functions  $f/g$ . Here we provide a direct proof that also works simultaneously for all fields  $K$ .

**Theorem 3.9.** Take  $K, f, g, \sigma : X_K \rightarrow U$  and  $\alpha_\Phi(\sigma, D_K), \beta_\Phi(\sigma, D_K)$  as in Section 2.

(1) Assume that  $\beta_\Phi(\sigma, D_K) \neq -\infty$ , and that it is equal to  $-\frac{v_i}{N_i}$  precisely for  $i \in T_\beta (\subset T_+)$ . If  $\Phi$  is positive with  $\Phi(P) > 0$  for some  $P \in \sigma(\cup_{i \in T_\beta} E_i)$ , then  $\beta_\Phi(\sigma, D_K)$  is a pole of  $Z_\Phi(s; f/g)$ .

(2) Assume that  $\alpha_\Phi(\sigma, D_K) \neq +\infty$ , and that it is equal to  $\frac{v_i}{|N_i|}$  precisely for  $i \in T_\alpha (\subset T_-)$ . If  $\Phi$  is positive with  $\Phi(P) > 0$  for some  $P \in \sigma(\cup_{i \in T_\alpha} E_i)$ , then  $\alpha_\Phi(\sigma, D_K)$  is a pole of  $Z_\Phi(s; f/g)$ .

(3) Hence  $\alpha_\Phi(\sigma, D_K)$  and  $\beta_\Phi(\sigma, D_K)$  do not depend on  $(\sigma, D_K)$ .

*Proof.* (1) Take a generic point  $b$  in a component  $E$  of  $\sigma^{-1}(D_K)$  with numerical datum  $(N = N_f - N_g, v)$  such that  $\beta = \beta_\Phi(\sigma, D_K) = -\frac{v}{N}$  and  $\Phi(\sigma(b)) > 0$ . Take also a small enough polydisc-chart  $B \subset X_K$  around  $b$  with coordinates  $y = (y_1, \dots, y_n)$  such that  $\Phi(\sigma(P)) > 0$  for all  $P$  in the closure  $\bar{B}$  of  $B$ ,

$$(3.3) \quad \left( \frac{f}{g} \circ \sigma \right) (y) = \varepsilon(y) y_1^N \quad \text{and} \quad \sigma^*(dx_1 \wedge \dots \wedge dx_n) = \eta(y) y_1^{v-1},$$

with  $\varepsilon(y)$  and  $\eta(y)$  invertible power series on  $B$ . We can change the coordinate  $y_1$  in order to have that  $\varepsilon$  is a constant, and assume that  $|\eta(y)|_K$  is bounded below by a positive constant  $C_1$ . When  $K$  is a  $p$ -field, we could assume that  $|\eta(y)|_K$  is constant on  $B$ . Denoting  $C_2 = |\varepsilon|_K \min_{Q \in \sigma(\bar{B})} \Phi(Q)$ , we have on  $B$  that

$$(\Phi \circ \sigma)(y) \left| \left( \frac{f}{g} \circ \sigma \right) (y) \right|_K^\beta |\eta(y)|_K |y_1^{v-1}|_K \geq C_2 |y_1|_K^{N\beta} C_1 |y_1|_K^{v-1} = C_2 C_1 |y_1|_K^{-1}.$$

Then, by the  $K$ -analytic change of variables formula (see e.g. [31, Proposition 7.4.1] for  $p$ -fields),

$$\begin{aligned} \int_{U \setminus D_K} \Phi(x) \left| \frac{f(x)}{g(x)} \right|_K^\beta |dx|_K &= \int_{X_K \setminus \sigma^{-1}D_K} (\Phi \circ \sigma)(y) \left| \left( \frac{f}{g} \circ \sigma \right) (y) \right|_K^\beta |\sigma^* dx|_K \\ &\geq C_2 C_1 \int_{B \setminus \{y_1=0\}} |y_1|_K^{-1} |dy|_K = +\infty. \end{aligned}$$

Note that  $Z_\Phi(\gamma; f/g)$  exists and is positive for  $\gamma \in \mathbb{R}$  satisfying  $\beta < \gamma < 0$ . Then, by using the Monotone Convergence Theorem, we have that

$$\lim_{\gamma \rightarrow \beta} Z_\Phi(\gamma; f/g) = \lim_{\gamma \rightarrow \beta} \int_{U \setminus D_K} \Phi(x) \left| \frac{f(x)}{g(x)} \right|_K^\gamma |dx|_K = +\infty.$$

Hence  $\beta$  must be a pole of  $Z_\Phi(s; f/g)$ , being a meromorphic function in the whole complex plane.

(2) By Part (1) we have that  $-\alpha_\Phi(\sigma, D_K)$  is a pole of  $Z_\Phi(s; g/f)$ , and hence  $\alpha_\Phi(\sigma, D_K)$  is a pole of  $Z_\Phi(s; f/g) = Z_\Phi(-s; g/f)$ .  $\square$

**Remark 3.10.** Note that, whenever  $T$  is finite, in particular when  $f$  and  $g$  are polynomials, the global  $\alpha$  and  $\beta$  of Definition 2.7(2) also do not depend on  $\Phi$ .

**Remark 3.11.** In [42, Theorem 2.7] we showed in the context of zeta functions of mappings over  $p$ -fields that the analogue of  $\beta$  is a real part of a pole. With the technique above one can in fact show that also in that context  $\beta$  itself is a pole.

**Corollary 3.12.** Take  $\frac{f}{g} : U \setminus D_K \rightarrow K$  as in Section 2.

(1) Assume that there exists a point  $x_0 \in U$  such that  $f(x_0) = 0$  and  $g(x_0) \neq 0$ . Then, for any positive test function  $\Phi$  with support in a small enough neighborhood of  $x_0$ , the zeta function  $Z_\Phi(s; f/g)$  has a negative pole.

(2) Assume that there exists a point  $x_0 \in U$  such that  $f(x_0) \neq 0$  and  $g(x_0) = 0$ . Then, for any positive test function  $\Phi$  with support in a small enough neighborhood of  $x_0$ , the zeta function  $Z_\Phi(s; f/g)$  has a positive pole.

(3) In particular, if  $K = \mathbb{C}$  and  $f$  and  $g$  are polynomials, then  $Z_\Phi(s; f/g)$  always has a pole for an appropriate positive test function  $\Phi$ .

*Proof.* Given  $x_0$  as in (1), there must exist a component  $E_i$  of  $\sigma^{-1}(D_K)$  in an embedded resolution  $\sigma$  of  $D_K$  for which  $N_i > 0$ . Hence  $\beta(\sigma, D_K) \neq -\infty$  and then, by Theorem 3.9, it is a pole. Alternatively,  $f/g$  is analytic in a neighborhood of  $x_0$ , and hence this is really just the classical result.

The argument for (2) is similar.  $\square$

When  $f$  and  $g$  have exactly the same zeroes in  $U$ , we cannot derive any conclusion from Corollary 3.12. In fact, in that case all possibilities can occur: no poles, only positive poles, only negative poles, or both positive and negative poles. We provide an example for each situation.

**Example 3.13.** Let  $K = \mathbb{R}$ , or  $K = \mathbb{Q}_p$  such that  $-1$  is not a square in  $\mathbb{Q}_p$  (that is,  $p \equiv 3 \pmod{4}$ ). In the examples below we take  $U = K^2$  and we consider in each case polynomials  $f$  and  $g$  such that the zero locus over  $K$  of both  $f$  and  $g$  consists of the origin.

(1) Take  $f = (x^2 + y^2)^2$  and  $g = x^4 + y^4$ . An embedded resolution of  $D_K$  is obtained by blowing up at the origin; the exceptional curve has datum  $(N, v) = (0, 2)$ . Hence, for any test function  $\Phi$ ,  $Z_\Phi(s; f/g)$  does not have poles.

(2) Take  $f = x^2 + y^2$  and  $g = x^4 + y^4$ . Again an embedded resolution of  $D_K$  is obtained by blowing up at the origin; the exceptional curve now has datum  $(N, v) = (-2, 2)$ . Hence 1 is a pole of  $Z_\Phi(s; f/g)$  when  $\Phi$  is positive around the origin, and there are no negative (real parts of) poles.

(3) Take  $f = x^4 + y^4$  and  $g = x^2 + y^2$ . Analogously,  $-1$  is a pole of  $Z_\Phi(s; f/g)$  when  $\Phi$  is positive around the origin, and there are no positive (real parts of) poles.

(4) Take  $f = y^2 + x^4$  and  $g = x^2 + y^4$ . Now an embedded resolution of  $D_K$  is obtained by first blowing up at the origin, yielding an exceptional curve with datum  $(0, 2)$ , and next performing two blow-ups at centres the origins of the two charts, yielding exceptional curves with data  $(2, 3)$  and  $(-2, 3)$ . Hence  $-3/2$  and  $3/2$  are poles of  $Z_\Phi(s; f/g)$  when  $\Phi$  is positive around the origin.

**Lemma 3.14.** Take  $\frac{f}{g} : U \setminus D_K \rightarrow K$  as in Section 2.

(1) The zeta function  $Z_\Phi(s; f/g)$  has a negative pole, for an appropriate positive  $\Phi$ , if and only if there exists a bounded sequence  $(a_n)_{n \in \mathbb{Z}_{\geq 0}} \subset U \setminus D_K$  such that

$$(3.4) \quad \lim_{n \rightarrow \infty} \left| \frac{f(a_n)}{g(a_n)} \right|_K = 0.$$

(2) The zeta function  $Z_\Phi(s; f/g)$  has a positive pole, for an appropriate positive  $\Phi$ , if and only if there exists a bounded sequence  $(a_n)_{n \in \mathbb{Z}_{\geq 0}} \subset U \setminus D_K$  such that

$$(3.5) \quad \lim_{n \rightarrow \infty} \left| \frac{f(a_n)}{g(a_n)} \right|_K = \infty.$$

*Proof.* (1) We consider an adapted embedded resolution  $\sigma$  of  $D_K$  as in Theorem 2.2, obtained by resolving the indeterminacies of  $f/g$ . From Theorem 3.9, it is clear that  $Z_\Phi(s; f/g)$  has a negative pole for an appropriate positive  $\Phi$  if and only if there exists a component  $E_i$  of  $\sigma^{-1}(D_K)$  with  $N_i > 0$ .

Assume first that  $N_i > 0$  for the component  $E_i$ . Take a generic point  $b$  of  $E_i$  and a small enough chart  $B \subset X_K$  around  $b$  with coordinates  $y = (y_1, \dots, y_n)$ , such that the equation of  $E_i$  around  $b$  is  $y_1 = 0$ . Take a sequence  $(a'_n)_{n \in \mathbb{Z}_{\geq 0}}$  defined by  $a'_n = (z_n, 0, \dots, 0)$ , where all  $z_n \neq 0$  and  $\lim_{n \rightarrow \infty} |z_n|_K = 0$ . Clearly  $a'_n$  converges (in norm) to  $b$ , and hence  $\frac{f}{g}(\sigma(a'_n)) = (\frac{f}{g} \circ \sigma)(a'_n)$  converges to  $(\frac{f}{g} \circ \sigma)(b)$ , and this equals zero since  $N_i > 0$ . Consequently the sequence  $(\sigma(a'_n))_{n \in \mathbb{Z}_{\geq 0}}$  is as desired.

Conversely, assume that the bounded sequence  $(a_n)_{n \in \mathbb{Z}_{\geq 0}} \subset U \setminus D_K$  satisfies  $\lim_{n \rightarrow \infty} \left| \frac{f(a_n)}{g(a_n)} \right|_K = 0$ . Say the  $a_n$  belong to the compact subset  $A$  of  $U$ . Since  $\sigma$  is an isomorphism outside the inverse image of  $D_K$ , each  $a_n$  has a unique preimage

$a'_n$  in  $X_K$ . The  $a'_n$  belong to the compact set  $\sigma^{-1}(A)$  (recall that  $\sigma$  is proper), and hence we may assume, maybe after restricting to a subsequence, that the sequence  $(a'_n)$  converges (in norm) to an element  $a'$  of  $\sigma^{-1}(A)$ . The map  $\frac{f}{g} \circ \sigma$  is everywhere defined (as map to the projective line) and continuous, yielding

$$\left(\frac{f}{g} \circ \sigma\right)(a') = \left(\frac{f}{g} \circ \sigma\right)\left(\lim_{n \rightarrow \infty} a'_n\right) = \lim_{n \rightarrow \infty} \left(\frac{f}{g} \circ \sigma\right)(a'_n) = \lim_{n \rightarrow \infty} \frac{f}{g}(a_n) = 0.$$

Consequently  $a'$  belongs to the union of the  $E_i$  satisfying  $N_i > 0$ , and then in particular there is at least one such  $E_i$ .

(2) This is proved analogously.  $\square$

**3.4. Denef's explicit formula.** Here we assume that  $f$  and  $g$  are polynomials over a  $p$ -field  $K$ , and that  $\Phi$  is the characteristic function of a union  $W$  of cosets mod  $P_K^n$  in  $R_K^n$ . Denef introduced in [15] the notion of *good reduction* mod  $P_K$  for an embedded resolution, and proved an explicit formula [15, Theorem 3.1] for  $Z_\Phi(s; f)$  when  $\sigma$  has good reduction mod  $P_K$ . His arguments extend to the case of a rational function  $f/g$ .

Before stating the formula, we mention the following important property. When  $f$  and  $g$  are defined over a number field  $F$ , one can choose also the embedded resolution  $\sigma$  of  $f^{-1}\{0\} \cup g^{-1}\{0\}$  to be defined over  $F$ . We can then consider  $f/g$  and  $\sigma$  over any non-archimedean completion  $K$  of  $F$ , and then  $\sigma$  has good reduction mod  $P_K$  for all but a finite number of completions  $K$ .

**Theorem 3.15.** *Let  $f, g \in R_K[x_1, \dots, x_n]$  such that  $f/g$  is not constant, and suppose that  $\Phi$  is the characteristic function of  $W$  as above. If  $\sigma$  has good reduction mod  $P_K$ , then, with the notation of Theorem 2.2, we have*

$$Z_\Phi(s; f/g) = q^{-n} \sum_{I \subset T} c_I \prod_{i \in I} \frac{q-1}{q^{v_i + N_i s} - 1},$$

where

$$c_I = \text{card} \{a \in \overline{X}(\overline{K}) \mid a \in \overline{E_i}(\overline{K}) \Leftrightarrow i \in I; \text{ and } \overline{\sigma}(a) \in \overline{W}\}.$$

Here  $\overline{\phantom{x}}$  denotes the reduction mod  $P_K$ , for which we refer to [15, Section 2].

**Example 3.16.** We compute  $Z(s) := \int_{(p\mathbb{Z}_p)^2} \left| \frac{f(x,y)}{g(x,y)} \right|_{\mathbb{Q}_p}^s |dx \wedge dy|_{\mathbb{Q}_p}$  for the rational functions  $f/g$  in Example 3.13, using Theorem 3.15. Recall that  $p \equiv 3 \pmod{4}$ .

In case (1) we have simply that  $Z(s) = \frac{1}{p^2}$ , but this was already obvious from the defining integral, since we are just integrating the constant function 1. The cases (2) and (3) yield

$$Z(s) = \frac{p^2 - 1}{p^2(p^{2-2s} - 1)} \quad \text{and} \quad Z(s) = \frac{p^2 - 1}{p^2(p^{2+2s} - 1)},$$

respectively. For case (4) we obtain

$$\begin{aligned} Z(s) &= \frac{1}{p^2} \left[ p \frac{p-1}{p^{3+2s}-1} + p \frac{p-1}{p^{3-2s}-1} + (p-1) \frac{p-1}{p^2-1} \right. \\ &\quad \left. + \frac{(p-1)^2}{(p^{3+2s}-1)(p^2-1)} + \frac{(p-1)^2}{(p^{3-2s}-1)(p^2-1)} \right] \\ &= \frac{(p-1)(p^{4+2s} + p^{4-2s} + p^5 - p^4 + p^3 - p^2 - p - 1)}{p^2(p^{3+2s}-1)(p^{3-2s}-1)}. \end{aligned}$$

In addition, we provide an expression for case (4) when  $p \equiv 1 \pmod{4}$ . Then the strict transform of  $f^{-1}\{0\}$  and  $g^{-1}\{0\}$  (with data  $(1, 1)$  and  $(-1, 1)$ , respectively), intersects the second and third exceptional curve, respectively, in two points. The formula of Theorem 3.15 now yields nine terms, and we obtain after an elementary computation that

$$Z(s) = \frac{(p-1)D(p^s, p)}{p^2(p^{3+2s}-1)(p^{3-2s}-1)(p^{1+s}-1)(p^{1-s}-1)}$$

with

$$\begin{aligned} D(p^s, p) &= p^{5+3s} + p^{5-3s} + (p^2 - 2p - 1)(p^{4+2s} + p^{4-2s}) \\ &\quad + (-p^5 + p^3 + p^2 - p - 1)(p^{1+s} + p^{1-s}) + p^7 - p^6 + 2p^5 - 2p^4 + 2p^2 + 3p - 1. \end{aligned}$$

**3.5. Motivic and topological zeta functions.** The analogue of the original explicit formula of Denef plays an important role in the study of the motivic zeta function associated to a regular function [19]. One can associate more generally a motivic zeta function to a rational function on a smooth variety, and obtain a similar formula for it in terms of an embedded resolution.

Since this is not the focus of the present paper, we just formulate the more general definition and formula, referring to e.g. [20], [41] for the notion of jets and arcs, Grothendieck ring, and motivic measure. Set as usual  $[\cdot]$  for the class of a variety in the Grothendieck ring of algebraic varieties over a field  $F$ , and  $\mathbb{L} := [\mathbb{A}^1]$ . We also denote by  $\mathcal{M}$  the localization of that Grothendieck ring with respect to  $\mathbb{L}$  and the elements  $\mathbb{L}^a - 1, a \in \mathbb{Z}_{>0}$ .

Let  $Y$  be a smooth algebraic variety of dimension  $n$  over a field  $F$  of characteristic zero, and  $f/g$  a non-constant rational function on  $Y$ . Let  $W$  be a subvariety of  $Y$ . Denote for  $n \in \mathbb{Z}$  by  $\mu(\mathfrak{X}_n^W)$  the motivic measure of the arcs  $\gamma$  on  $Y$  with origin in  $W$  for which  $\text{ord}_t(\gamma^*(f/g)) = n$ . A priori, it is not clear that this set of arcs is measurable, but one can show that  $\mu(\mathfrak{X}_n^W)$  is well defined in  $\mathcal{M}$  (see [12, Theorem 10.1.1]).

**Definition 3.17.** With notation as above, the *motivic zeta function* associated to  $f/g$  (and  $W$ ) is

$$Z_W(f/g; T) = \sum_{n \in \mathbb{Z}} \mu(\mathfrak{X}_n^W) T^n.$$

Note that this is a series in  $T$  and  $T^{-1}$  (over  $\mathcal{M}$ ), and that such expressions in general do not form a ring. As in the classical case however,  $Z_W(f/g; T)$  turns out to be a rational function in  $T$ , see [12, Theorem 5.7.1].

Using the change of variables formula in motivic integration ([20, Lemma 3.3], [12, Theorem 12.1.1]), one obtains the following formula for  $Z_W(f/g; T)$  in terms of an embedded resolution of  $f^{-1}\{0\} \cup g^{-1}\{0\}$ , which generalizes the similar formula in the classical case.

**Theorem 3.18.** *Let  $\sigma : X \rightarrow Y$  be an embedded resolution of  $f^{-1}\{0\} \cup g^{-1}\{0\}$ , for which we use the analogous notation  $E_i$ ,  $N_i = N_{f,i} - N_{g,i}$  and  $v_i$  ( $i \in T$ ) as in Theorem 2.2. With also  $E_I^\circ := (\cap_{i \in I} E_i) \setminus (\cup_{k \notin I} E_k)$  for  $I \subset T$ , we have*

$$Z_W(f/g; T) = \sum_{I \subset T} [E_I^\circ \cap \sigma^{-1}W] \prod_{i \in I} \frac{(\mathbb{L} - 1) T^{N_i}}{\mathbb{L}^{v_i} - T^{N_i}}.$$

Note that above we use only very special cases of the far reaching general theory in [12]. Alternatively, one could obtain Theorem 3.18 by considering a classical two-variable motivic zeta function associated to  $f$  and  $g$ , the change of variables formula, and a specialization argument as in the proof of Theorem 3.2.

We also want to mention that Raibaut [39, Section 4.1] introduced another kind of motivic zeta function for a rational function, in order to define a motivic Milnor fibre at the germ of an indeterminacy point.

Specializing to topological Euler characteristics, denoted by  $\chi(\cdot)$ , as in [19, (2.3)] or [41, (6.6)], we obtain the expression

$$Z_{top,W}(f/g; s) := \sum_{I \subset T} \chi(E_I^\circ \cap \sigma^{-1}W) \prod_{i \in I} \frac{1}{v_i + N_i s} \in \mathbb{Q}(s),$$

which is then independent of the chosen embedded resolution. (When the base field is not the complex numbers, we consider  $\chi(\cdot)$  in étale  $\overline{\mathbb{Q}_\ell}$ -cohomology as in [19].) It can be taken as a definition for the *topological zeta function* associated to  $f/g$  (and  $W$ ), generalizing the original one of Denef and Loeser associated to a polynomial [18].

For  $n = 2$ , Gonzalez Villa and Lemahieu give in [24] a different definition of a topological zeta function associated to a meromorphic function  $f/g$ , roughly not taking into account the components  $E_i$  with  $N_i < 0$ . That construction is more ad hoc, but well suited to generalize the so-called *monodromy conjecture* in dimension 2 to meromorphic functions.

#### 4. OSCILLATORY INTEGRALS

In this section we start the study of the asymptotic behavior of oscillatory integrals attached to meromorphic functions. In the next section we then prove our main results.

##### 4.1. The set-up.

4.1.1. *Additive characters.* We denote

$$\Psi(x) := \begin{cases} \exp(2\pi\sqrt{-1}x) & \text{if } x \in K = \mathbb{R} \\ \exp(4\pi\sqrt{-1}\operatorname{Re}(x)) & \text{if } x \in K = \mathbb{C}. \end{cases}$$

We call  $\Psi$  the *standard additive character on  $K = \mathbb{R}, \mathbb{C}$* . Given

$$z = \sum_{n=n_0}^{\infty} z_n p^n \in \mathbb{Q}_p, \text{ with } z_n \in \{0, \dots, p-1\} \text{ and } z_{n_0} \neq 0,$$

we set

$$\{z\}_p := \begin{cases} 0 & \text{if } n_0 \geq 0 \\ \sum_{n=n_0}^{-1} z_n p^n & \text{if } n_0 < 0, \end{cases}$$

the *fractional part of  $z$* . We also put  $\{0\}_p = 0$ . Then  $\exp(2\pi\sqrt{-1}\{z\}_p)$ ,  $z \in \mathbb{Q}_p$ , determines an additive character on  $\mathbb{Q}_p$ , trivial on  $\mathbb{Z}_p$  but not on  $p^{-1}\mathbb{Z}_p$ .

For a finite extension  $K$  of  $\mathbb{Q}_p$ , we recall that there exists an integer  $d \geq 0$  such that  $\operatorname{Tr}_{K/\mathbb{Q}_p}(z) \in \mathbb{Z}_p$  for  $|z|_K \leq q^d$ , but  $\operatorname{Tr}_{K/\mathbb{Q}_p}(z_0) \notin \mathbb{Z}_p$  for some  $z_0$  with  $|z_0|_K = q^{d+1}$ . The integer  $d$  is called the *exponent of the different* of  $K/\mathbb{Q}_p$ . It is

known that  $d \geq e - 1$ , where  $e$  is the ramification index of  $K/\mathbb{Q}_p$ , see e.g. [44, VIII Corollary of Proposition 1]. The additive character

$$\Psi(z) := \exp\left(2\pi\sqrt{-1}\left\{Tr_{K/\mathbb{Q}_p}(\mathfrak{p}^{-d}z)\right\}_p\right), \quad z \in K,$$

is a *standard additive character* of  $K$ , i.e.,  $\Psi$  is trivial on  $R_K$  but not on  $P_K^{-1}$ .

**4.1.2. Oscillatory integrals.** We take  $\frac{f}{g} : U \rightarrow K$  and  $\Phi$  as in Section 2. The oscillatory integral attached to  $\left(\frac{f}{g}, \Phi\right)$  is defined by

$$E_\Phi(z; f/g) = \int_{U \setminus D_K} \Phi(x) \Psi\left(z \cdot \frac{f(x)}{g(x)}\right) |dx|_K$$

for  $z \in K$ . Our goal is to study the asymptotic behavior of  $E_\Phi(z; f/g)$  as  $|z|_K \rightarrow +\infty$  and of  $E_\Phi(z; f/g) - \int_U \Phi(x) |dx|_K$  as  $|z|_K \rightarrow 0$ .

**4.2. Classical results.** In the classical case of an analytic function  $f$  (when  $g = 1$ ), the oscillatory integral above is related to the zeta function  $Z_\Phi(\omega; f)$  and to the fibre integral

$$F_\Phi(t; f) = \int_{f(x)=t} \Phi(x) \left| \frac{dx}{df} \right|_K$$

for  $t \in K^\times$ , where  $\frac{dx}{df}$  denotes the Gel'fand-Leray differential form on  $f(x) = t$ . More precisely, denoting by  $C_f := \{z \in U \mid \nabla f(z) = 0\}$  the critical set of  $f$ , one assumes that  $C_f \cap \text{supp } \Phi \subset f^{-1}\{0\}$ . Then asymptotic expansions of  $E_\Phi(z; f)$  as  $|z|_K \rightarrow +\infty$  are closely related to asymptotic expansions of  $F_\Phi(t; f)$  as  $|t|_K \rightarrow 0$ , and to poles of  $Z_\Phi(\omega; f)$ . These matters were extensively studied by many authors, for instance in several papers of Barlet in the case of  $\mathbb{R}$ -fields. See also [2] when  $K = \mathbb{R}$ . Igusa presented in [29] a uniform theory over arbitrary local fields of characteristic zero (for polynomials  $f$ , but also valid for analytic functions).

We first recall three spaces of functions from [29] that we will use, and the relations between them.

**4.2.1. Relating spaces of functions through Mellin and Fourier transform.**

**Definition 4.1.** Let  $K$  be an  $\mathbb{R}$ -field. Let  $\Lambda$  be a strictly increasing (countable) sequence of positive real numbers with no finite accumulation point, and  $\{n_\lambda\}_{\lambda \in \Lambda}$  a sequence of positive integers.

(1) We define  $\mathcal{G}$  as the space of all complex-valued functions  $G$  on  $K^\times$  such that

- (i)  $G \in C^\infty(K^\times)$ ;
- (ii)  $G(x)$  behaves like a Schwartz function of  $x$  as  $|x|_K$  tends to infinity;
- (iii) we have the asymptotic expansion

$$(4.1) \quad G(x) \approx \sum_{\lambda \in \Lambda} \sum_{m=1}^{n_\lambda} b_{\lambda,m} \left( \frac{x}{|x|} \right) |x|_K^{\lambda-1} (\ln |x|_K)^{m-1} \quad \text{as } |x|_K \rightarrow 0,$$

where the  $b_{\lambda,m}$  are smooth functions on  $\{u \in K^\times \mid |u|_K = 1\}$ . In addition, this expansion is termwise differentiable and uniform in  $x/|x|$  (we refer to [29, page 23] for a detailed explanation of this terminology).

(2) We define  $\mathcal{Z}$  as the space of all complex-valued functions  $Z$  on  $\Omega(K^\times)$  such that



- (i)  $Z(\omega = \omega_s \chi_l(ac))$  is meromorphic on  $\Omega(K^\times)$  with poles at most at  $s \in -\Lambda$ ;
- (ii) there are constants  $r_{\lambda, m, l}$  such that  $Z(\omega_s \chi_l(ac)) - \sum_{m=1}^{n_\lambda} \frac{r_{\lambda, m, l}}{(s+\lambda)^m}$  is holomorphic for  $s$  close enough to  $-\lambda$ , for all  $\lambda \in \Lambda$ ;
- (iii) for every polynomial  $P \in \mathbb{C}[s, l]$  and every vertical strip in  $\mathbb{C}$  of the form  $B_{\sigma_1, \sigma_2} := \{s \in \mathbb{C} \mid \operatorname{Re}(s) \in (\sigma_1, \sigma_2)\}$ , we have that  $P(s, l)Z(\omega_s \chi_l(ac))$  is uniformly bounded for all  $l$  and for  $s \in B_{\sigma_1, \sigma_2}$  with neighborhoods of the points in  $-\Lambda$  removed therefrom.

**Definition 4.2.** Let  $K$  be a  $p$ -field. Let  $\Lambda$  be a finite set of complex numbers mod  $2\pi i / \ln q$  with positive real part, and  $\{n_\lambda\}_{\lambda \in \Lambda}$  a sequence of positive integers.

(1) We define  $\mathcal{G}$  as the space of all complex-valued functions  $G$  on  $K^\times$  such that

- (i)  $G$  is locally constant;
- (ii)  $G(x) = 0$  when  $|x|_K$  is sufficiently large;
- (iii) we have the expansion

$$(4.2) \quad G(x) = \sum_{\lambda \in \Lambda} \sum_{m=1}^{n_\lambda} b_{\lambda, m}(ac(x)) |x|_K^{\lambda-1} (\ln |x|_K)^{m-1} \quad \text{for sufficiently small } |x|_K,$$

where the  $b_{\lambda, m}$  are locally constant functions on  $R_K^\times$ .

(2) We define  $\mathcal{Z}$  as the space of all complex-valued functions  $Z$  on  $\Omega(K^\times)$  such that

- (i) there are constants  $r_{\lambda, m, \chi}$  such that  $Z(\omega_s \chi(ac)) - \sum_{\lambda \in \Lambda} \sum_{m=1}^{n_\lambda} \frac{r_{\lambda, m, \chi}}{(1-q^{-\lambda} q^{-s})^m}$  is a Laurent polynomial in  $q^{-s}$ ;
- (ii) there exists a positive integer  $e$  such that  $Z(\omega_s \chi(ac))$  is identically zero unless the conductor  $e_\chi$  of  $\chi$  satisfies  $e_\chi \leq e$ .

In order to simplify notation in the sequel of this article, we adapted slightly Igusa's setup in the sense that we consider expansions involving  $|x|_K^{\lambda-1}$  instead of  $|x|_K^\lambda$  in (4.1) and (4.2). This implies that the statements of the theorems below differ also slightly from the formulation in [29].

**Theorem 4.3.** [29, I Theorems 4.2, 4.3 and 5.3] *The Mellin transform  $M_K$  induces a bijective correspondence between the spaces  $|x|_K \mathcal{G}$  and  $\mathcal{Z}$ . More precisely, for any  $G \in \mathcal{G}$ , define  $M_K(|x|_K G)$  as a function on  $\Omega_{(0, +\infty)}(K^\times)$  by*

$$M_K(|x|_K G)(\omega) = \int_{K^\times} |x|_K G(x) \omega(x) \frac{|dx|_K}{m_K |x|_K},$$

where  $m_K$  is 2,  $2\pi$  or  $1 - q^{-1}$ , according as  $K$  is  $\mathbb{R}$ ,  $\mathbb{C}$ , or a  $p$ -field. Then its meromorphic continuation to  $\Omega(K^\times)$  is in  $\mathcal{Z}$ , and the map  $G \mapsto M_K(|x|_K G)$  induces a bijection between the spaces  $|x|_K \mathcal{G}$  and  $\mathcal{Z}$ .

**Remark 4.4.** Under the bijective correspondence  $|x|_K G \mapsto Z$  above, the coefficients in the expansion of  $|x|_K G$  and the coefficients in the Laurent expansion of  $Z$  around its poles determine each other. See [29, I Theorems 4.2, 4.3 and 5.3] for the exact relations.

**Theorem 4.5.** [29, II Theorem 2.1] (1) *Let  $K$  be an  $\mathbb{R}$ -field. Let  $\Lambda$  be a strictly increasing (countable) sequence of positive real numbers with no finite accumulation point, and  $\{n_\lambda\}_{\lambda \in \Lambda}$  a sequence of positive integers. Then the space  $\mathcal{G}^*$  of Fourier transforms of functions in  $\mathcal{G}$  consists precisely of all complex-valued functions  $H$  on  $K$  such that*

- (i)  $H \in C^\infty(K)$ ;  
(ii) we have the termwise differentiable uniform asymptotic expansion

$$(4.3) \quad H(x) \approx \sum_{\lambda \in \Lambda} \sum_{m=1}^{n_\lambda} c_{\lambda,m} \left( \frac{x}{|x|} \right) |x|_K^{-\lambda} (\ln |x|_K)^{m-1} \text{ as } |x|_K \rightarrow \infty,$$

where the  $c_{\lambda,m}$  are smooth functions on  $\{u \in K^\times \mid |u|_K = 1\}$ . Furthermore, if  $H$  is the Fourier transform  $G^*$  of  $G \in \mathcal{G}$  with asymptotic expansion (4.1), then (4.3) is the termwise Fourier transform of (4.1).

(2) Let  $K$  be a  $p$ -field. Let  $\Lambda$  be a finite set of complex numbers mod  $2\pi i / \ln q$  with positive real part, and  $\{n_\lambda\}_{\lambda \in \Lambda}$  a sequence of positive integers. Then the space  $\mathcal{G}^*$  of Fourier transforms of functions in  $\mathcal{G}$  consists precisely of all complex-valued functions  $H$  on  $K$  such that

- (i)  $H$  is locally constant;  
(ii) we have the expansion

$$(4.4) \quad H(x) = \sum_{\lambda \in \Lambda} \sum_{m=1}^{n_\lambda} c_{\lambda,m}(ac(x)) |x|_K^{-\lambda} (\ln |x|_K)^{m-1} \text{ for sufficiently large } |x|_K,$$

where the  $c_{\lambda,m}$  are locally constant functions on  $R_K^\times$ . Furthermore, if  $H$  is the Fourier transform  $G^*$  of  $G \in \mathcal{G}$  satisfying (4.2), then (4.4) is the termwise Fourier transform of (4.2).

(3) For all  $K$  the inverse of the map  $\mathcal{G} \rightarrow \mathcal{G}^* : G \mapsto G^*$  is given by the ‘generalized Fourier transform’  $H \mapsto H^*(-x)$ , where

$$H^*(x) = \lim_{r \rightarrow \infty} \int_{|y|_K \leq r} H(y) \Psi(xy) |dy|_K.$$

In order to describe the relation between the coefficients  $b$  and  $c$  above, we first introduce some notation. Let  $\chi$  be a character of  $\{u \in K^\times \mid |u|_K = 1\}$ . We first assume that  $K$  is an  $\mathbb{R}$ -field. For  $\chi = \chi_l$ , we denote

$$w_\chi(s) = i^{|l|} (\pi [K : \mathbb{R}])^{\frac{1}{2}[K:\mathbb{R}](1-2s)} \frac{\Gamma\left(\frac{1}{2}[K:\mathbb{R}]s + \frac{|l|}{2}\right)}{\Gamma\left(\frac{1}{2}[K:\mathbb{R}](1-s) + \frac{|l|}{2}\right)},$$

where  $\Gamma(s)$  is the complex Gamma function. When  $K$  is a  $p$ -field, we put

$$w_\chi(s) = \frac{1 - q^{s-1}}{1 - q^{-s}} \quad \text{if } \chi = 1 \quad \text{and} \quad w_\chi(s) = g_\chi q^e \chi^s \quad \text{if } \chi \neq 1,$$

where  $e$  is the conductor of  $\chi$  and  $g_\chi$  is a (non-zero) Gaussian sum (see [29, page 57]).

**Proposition 4.6.** [29, II Remark 1 page 67] (1) For all  $\lambda$  and  $m$ , let  $b_{\lambda,m,\chi}$  be the Fourier coefficients of  $b_{\lambda,m}(u)$  in Definitions 4.1 and 4.2, that is, we have the expansion

$$b_{\lambda,m}(u) = \sum_{\chi} b_{\lambda,m,\chi} \chi(u),$$

and similarly

$$c_{\lambda,m}(u) = \sum_{\chi} c_{\lambda,m,\chi} \chi(u)$$

for  $c_{\lambda,m}(u)$  in Theorem 4.5. Here, when  $K$  is an  $\mathbb{R}$ -field,  $\chi$  runs over the  $\chi_l$  with  $l \in \{0, 1\}$  or  $l \in \mathbb{Z}$ , according as  $K$  is  $\mathbb{R}$  or  $\mathbb{C}$ , and when  $K$  is a  $p$ -field,  $\chi$  runs over the characters of  $R_K^\times$ .

These Fourier coefficients are related as follows:

$$(4.5) \quad c_{\lambda,m,\chi^{-1}} = (-1)^{m-1} \sum_{j=m}^{n_\lambda} \binom{j-1}{m-1} \left( \frac{d^{j-m}}{ds^{j-m}} w_\chi(\lambda) \right) b_{\lambda,j,\chi}.$$

Fixing  $\lambda$  and  $\chi$ , the equation (4.5) expresses, for  $1 \leq m \leq n_\lambda$ , the  $c_{\lambda,m,\chi^{-1}}$  linearly in terms of the  $b_{\lambda,m,\chi}$  with a coefficient matrix that is upper triangular of size  $n_\lambda$ , with everywhere  $w_\chi(\lambda)$  on the diagonal, and nonzero multiples of  $w'_\chi(\lambda)$  just above the diagonal.

(2) If  $w_\chi(\lambda) \neq 0$ , then the rank of that matrix is  $n_\lambda$  and then the  $b_{\lambda,m,\chi}$  can conversely be expressed in and are determined by the  $c_{\lambda,m,\chi^{-1}}$ .

On the other hand, if  $w_\chi(\lambda) = 0$ , then  $w'_\chi(\lambda) \neq 0$ , and hence the rank of that matrix is  $n_\lambda - 1$ . In this case,  $c_{\lambda,n_\lambda,\chi^{-1}} = 0$ , the coefficient  $b_{\lambda,1,\chi}$  does not appear in the expression for the  $c_{\lambda,m,\chi^{-1}}$ ,  $1 \leq m \leq n_\lambda - 1$ , and the  $b_{\lambda,m,\chi}$ ,  $2 \leq m \leq n_\lambda$ , can be expressed in and are determined by the  $c_{\lambda,m,\chi^{-1}}$ ,  $1 \leq m \leq n_\lambda - 1$ .

(3) For an  $\mathbb{R}$ -field  $K$  and  $\chi = \chi_l$ , we have that  $w_\chi(s) = 0$  if and only if  $s \in 1 + \frac{1}{[K:\mathbb{R}]}(l + 2\mathbb{Z}_{\geq 0})$ . For a  $p$ -field  $K$ , we have that  $w_\chi(s) = 0$  if and only if  $\chi = 1$  and  $s = 1 \pmod{2\pi i / \ln q}$ .

**Remark 4.7.** In the classical case of a polynomial/analytic function  $f$  (when  $g = 1$ ), an important result in [29] is that  $Z_\Phi(\omega; f)$  belongs to the space  $\mathcal{Z}$ , assuming that  $C_f \cap \text{supp } \Phi \subset f^{-1}\{0\}$ . Then Theorems 4.3 and 4.5 lead to the relations between asymptotic expansions of  $E_\Phi(z; f)$  as  $|z|_K \rightarrow +\infty$ , asymptotic expansions of  $F_\Phi(t; f)$  as  $|t|_K \rightarrow 0$ , and poles of  $Z_\Phi(\omega; f)$ .

Now for meromorphic functions  $f/g$ , the zeta function  $Z_\Phi(\omega; f/g)$  is in general *not* in  $\mathcal{Z}$ , and restricting the support of  $\Phi$  will not remedy that. As an illustration we indicate the problem when  $K$  is a  $p$ -field: in general  $Z_\Phi(\omega; f/g)$  is *not* identically zero when the conductor of  $\chi = \omega|_{R_K^\times}$  is large enough.

A crucial point in the proof of this result in the classical case is namely that an integral of the form  $\int_{p^e \mathbb{Z}_p} \chi(1+x) |dx|_{\mathbb{Q}_p}$  vanishes as soon as the conductor of  $\chi$  is large enough. In our case, due to the new feature in Remark 2.6, one also has to deal for instance with integrals of the form

$$\int_{p^e \mathbb{Z}_p} \chi(1+x) |x|_{\mathbb{Q}_p}^{\nu-1} |dx|_{\mathbb{Q}_p}$$

with  $\nu > 1$ . And those do *not* vanish for  $\chi$  with large enough conductor, as is shown by a straightforward computation.

In order to derive asymptotic expansions for  $E_\Phi(z; f/g)$ , we need a new strategy. We do use Theorems 4.3 and 4.5 in the sequel, but in an appropriate different setting.

**4.3. Set-up on the resolution space.** From now on we fix an adapted embedded resolution  $\sigma : X_K \rightarrow U$  of  $D_K$ , obtained by resolving the indeterminacies of the map  $f/g$ , we use the notation of Theorem 2.2 and the classical notation  $\sigma^* \Phi = \Phi \circ \sigma$ . We denote by  $\rho : X_K \rightarrow \mathbb{P}^1$  the *morphism*  $(f/g) \circ \sigma$ .

Analogously as in Remarks 3.3 and 3.6, we can describe

$$E_\Phi(z; f/g) = \int_{X_K \setminus \sigma^{-1}D_K} (\sigma^*\Phi)(y) \Psi(z \cdot \rho(y)) |\sigma^*dx|_K$$

as a complex linear combination of local contributions, described by ‘elementary’ oscillating integrals in local coordinates around appropriately chosen points  $b \in X_K$ . We first stress the following fact.

**Remark 4.8.** A point  $b \in X_K$  for which  $N_i = N_{f,i} - N_{g,i} > 0$  for at least one  $i \in \{1, \dots, r\}$ , satisfies  $\rho(b) = 0$ . Similarly, when  $N_i < 0$  for at least one  $i$ , we have that  $\rho(b) = \infty$ . And if all  $N_i = 0$ , then  $\rho(b) \in K^\times$ . As already mentioned in Remark 2.6, this last case also occurs for points  $b$  in  $\sigma^{-1}(D_K)$ .

These three possibilities for  $\rho(b)$  yield the following three types of local contributions. We use local coordinates  $y_1, \dots, y_n$  around  $b$ , and  $\Theta$  denotes the characteristic function of a small polydisc  $B$  around the origin, or a smooth function supported in the polydisc

$$B = \{y \in K^n \mid |y_j|_K < 1 \text{ for } j = 1, \dots, n\},$$

according as  $K$  is a  $p$ -field or an  $\mathbb{R}$ -field. When  $\rho(b) = 0$  and  $\rho(b) = \infty$ , we have respectively the types

$$(4.6) \quad E_1(z) := \int_{K^n} \Theta(y) \Psi\left(z \cdot c \prod_{i=1}^r y_i^{M_i}\right) \left(\prod_{i=1}^n |y_i|_K^{v_i-1}\right) |dy|_K$$

and

$$(4.7) \quad E_2(z) := \int_{K^n \setminus \cup_{i=1}^r \{y_i=0\}} \Theta(y) \Psi\left(z \cdot (c \prod_{i=1}^r y_i^{M_i})^{-1}\right) \left(\prod_{i=1}^n |y_i|_K^{v_i-1}\right) |dy|_K,$$

where  $r$ , all  $M_i$  and all  $v_i$  are positive integers (and  $r \leq n$ ), and  $c$  is a nonzero constant (that can be taken to be 1 if  $K = \mathbb{C}$ , and 1 or  $-1$  if  $K = \mathbb{R}$ ). Note that possibly some  $v_i = 1$ . When  $\rho(b) \in K^\times$ , we have the type

$$(4.8) \quad E_3(z) := \int_{K^n} \Theta(y) \Psi(z \cdot u(y)) \left(\prod_{i=1}^n |y_i|_K^{v_i-1}\right) |dy|_K,$$

where  $u(y)$  is invertible on the support of  $\Theta$ .

**4.4. Some auxiliary expansions.** In order to study

$$E_\Phi(z; f/g) = \int_{X_K \setminus \sigma^{-1}D_K} (\sigma^*\Phi)(y) \Psi(z \cdot \rho(y)) |\sigma^*dx|_K$$

via local contributions, we now fix an appropriate decomposition of  $\sigma^*\Phi$ .

We cover the (compact) support of  $\sigma^*\Phi$  by finitely many polydiscs  $B_j, j \in J$ , as above (all disjoint in the case of  $p$ -fields), making sure that the (compact) fibres  $\rho^{-1}\{0\} \cap \text{supp}(\sigma^*\Phi)$  and  $\rho^{-1}\{\infty\} \cap \text{supp}(\sigma^*\Phi)$  are completely covered by certain  $B_j, j \in J_0 \subset J$ , and  $B_j, j \in J_\infty \subset J$ , with centre  $b_j$  mapped by  $\rho$  to 0 and  $\infty$ , respectively. When  $K$  is a  $p$ -field, we define  $\Phi_0$  and  $\Phi_\infty$  as the restriction of  $\sigma^*\Phi$  to  $\cup_{j \in J_0} B_j$  and  $\cup_{j \in J_\infty} B_j$ , respectively. When  $K$  is an  $\mathbb{R}$ -field,  $\Phi_0$  and  $\Phi_\infty$  are these restrictions, modified by the partition of the unity that induces the functions  $\Theta$  on the local charts  $B_j$ .

We now define

$$E_0(z) = \int_{X_K \setminus \sigma^{-1}D_K} (\Phi_0)(y) \Psi(z \cdot \rho(y)) |\sigma^* dx|_K$$

and

$$E_\infty(z) = \int_{X_K \setminus \sigma^{-1}D_K} (\Phi_\infty)(y) \Psi(z \cdot \rho(y)) |\sigma^* dx|_K.$$

We first treat the case of oscillatory integrals over  $\mathbb{R}$ -fields.

**Proposition 4.9.** *Let  $K$  be an  $\mathbb{R}$ -field. We denote by  $m_\lambda$  the order of a pole  $\lambda$  of  $Z_\Phi(\omega; f/g)$ .*

(1) *Then  $E_0(z)$  has an asymptotic expansion of the form*

$$(4.9) \quad E_0(z) \approx \sum_{\lambda < 0} \sum_{m=1}^{m_\lambda} A_{\lambda,m} \left( \frac{z}{|z|} \right) |z|_K^\lambda (\ln |z|_K)^{m-1} \text{ as } |z|_K \rightarrow \infty,$$

where  $\lambda$  runs through all the negative poles of  $Z_\Phi(\omega; f/g)$  (for all  $\omega$ ), and each  $A_{\lambda,m}$  is a smooth function on  $\{u \in K^\times \mid |u|_K = 1\}$ .

(2) *Writing  $\omega = \omega_s \chi_l(ac)$ , we have that  $A_{\lambda,m_\lambda} = 0$  if  $|\lambda| \in 1 + \frac{1}{[K:\mathbb{R}]}(|l| + 2\mathbb{Z}_{\geq 0})$ .*

*Proof.* (1) We apply the ideas of Igusa's theory in [29, III] to the function  $\rho$  on  $X_K$  (in fact rather to the  $K$ -valued restriction of  $\rho$  to  $X_K \setminus \rho^{-1}\{\infty\}$ ) and the smooth function with compact support  $\Phi_0$ . Actually, Igusa formulates everything for polynomial functions, but his results are also valid for analytic functions on smooth manifolds, since his arguments are locally analytic on an embedded resolution space.

A crucial condition for his arguments to work is that  $C_\rho \cap \text{supp } \Phi_0 \subset \rho^{-1}\{0\}$ , where  $C_\rho$  denotes the critical locus of  $\rho$ . This condition is satisfied by our choice of  $\Phi_0$  and because  $\rho$  is locally monomial.

CLAIM. *The function  $Z_0$  on  $\Omega(K^\times)$ , defined by*

$$Z_0(\omega; \rho) = \int_{X_K \setminus \sigma^{-1}D_K} \Phi_0(y) \omega(\rho(y)) |\sigma^* dx|_K$$

for  $\omega \in \Omega_{(0,\infty)}(K^\times)$ , and extended to  $\Omega(K^\times)$  by meromorphic continuation, belongs to the class  $\mathcal{Z}$  of Definition 4.1.

*Proof of the claim.* The integral defining  $Z_0(\omega; \rho)$  is locally of the form

$$\int_{K^n} \Theta(y) \omega \left( c \prod_{i=1}^r y_i^{M_i} \right) \left( \prod_{i=1}^n |y_i|_K^{v_i-1} \right) |dy|_K.$$

This is precisely the local form that Igusa used to prove the analogue of the claim in [29, III §4], up to one important difference. In his case  $M_i > 0$  as soon as  $v_i > 1$ . In our case it is possible that  $M_i = 0$  while  $v_i > 1$ . This is however not a problem here because, in the polydiscs  $B_j, j \in J_0$ , that we consider, always  $M_i > 0$  for at least one  $i$ , and this is precisely what is needed in Igusa's argument.

We now relate in the usual way  $Z_0(\omega; \rho)$  to  $E_0(z)$  through the fibre integral

$$F_0(t) := \int_{\rho(y)=t} \Phi_0(y) \left| \frac{\sigma^* dx}{d\rho} \right|_K,$$

for  $t \in K^\times$ . We have that

$$Z_0(\omega; \rho) = \int_{K^\times} |t|_K F_0(t) \omega(t) \frac{|dt|_K}{|t|_K}$$

is the Mellin transform of  $m_K |t|_K F_0(t)$ , where  $m_K = 2$  for  $K = \mathbb{R}$  and  $m_K = 2\pi$  for  $K = \mathbb{C}$ . On the other hand,  $E_0(z) = \int_{K^\times} F_0(t) \Psi(z \cdot t) |dt|_K$  is the Fourier transform of  $F_0(t)$ .

Now, since  $Z_0(\omega; \rho)$  belongs to  $\mathcal{Z}$ , we have by Theorems 4.3 and 4.5 that the asymptotic expansion for  $E_0(z)$  at infinity is obtained from the asymptotic expansion of  $F_0(t)$  at zero, by computing its Fourier transform termwise. Here however the  $\lambda \in \Lambda$  are the poles of  $Z_0(\omega; \rho)$  (for all  $\omega$ ). But by the definition of  $\Phi_0$  and the explanation in Remark 3.3, these are precisely the negative poles of  $Z_\Phi(\omega; f/g)$  (for all  $\omega$ ).

(2) As in the classical case, this follows from [29, I Theorems 4.2 and 4.3] and Proposition 4.6.  $\square$

**Proposition 4.10.** *Let  $K$  be an  $\mathbb{R}$ -field; we put  $d_K := 1$  if  $K = \mathbb{R}$  and  $d_K := 1/2$  if  $K = \mathbb{C}$ . We denote by  $m_\lambda$  the order of a pole  $\lambda$  of  $Z_\Phi(\omega; f/g)$ . Then  $E_\infty(z)$  has an asymptotic expansion of the form*

$$(4.10) \quad E_\infty(z) - C \approx \sum_{\lambda > 0} \sum_{m=1}^{m_\lambda + \delta_\lambda} A_{\lambda, m} \left( \frac{z}{|z|} \right) |z|_K^\lambda (\ln |z|_K)^{m-1} \text{ as } |z|_K \rightarrow 0,$$

where  $C = E_\infty(0)$  is a constant,  $\lambda$  runs through  $d_K \mathbb{Z}_{>0}$  and all the positive poles of  $Z_\Phi(\omega; f/g)$  (for all  $\omega$ ) that are not in  $d_K \mathbb{Z}_{>0}$ ; furthermore  $\delta_\lambda = 0$  if  $\lambda \notin d_K \mathbb{Z}_{>0}$  and  $\delta_\lambda = 1$  if  $\lambda \in d_K \mathbb{Z}_{>0}$ . When  $\lambda$  is not a pole of  $Z_\Phi(\omega; f/g)$ , we put  $m_\lambda = 0$ . Finally, each  $A_{\lambda, m}$  is a smooth function on  $\{u \in K^\times \mid |u|_K = 1\}$ .

*Proof.* Firstly, we note that the positive poles of  $Z_\Phi(\omega; f/g)$  are precisely the poles of

$$\int_{X_K \setminus \sigma^{-1} D_K} \Phi_\infty(y) \omega(\rho(y)) |\sigma^* dx|_K = \int_{X_K \setminus \sigma^{-1} D_K} \Phi_\infty(y) \omega^{-1}((1/\rho)(y)) |\sigma^* dx|_K.$$

We consider  $1/\rho$  as a  $K$ -valued function on  $X_K \setminus \rho^{-1}\{0\}$ , and we apply the intermediate results in the proof of Proposition 4.9 to it. More precisely, replacing in that proof  $\rho$  by  $1/\rho$ ,  $\Phi_0$  by  $\Phi_\infty$  and  $\omega$  by  $\omega^{-1}$ , we derive that the function

$$F^\#(t) := \int_{(1/\rho)(y)=t} \Phi_\infty(y) \left| \frac{\sigma^* dx}{d(1/\rho)} \right|_K,$$

for  $t \in K^\times$ , belongs to the class  $\mathcal{G}$ , and that it is related by the Mellin transformation to the function  $Z_\infty$  on  $\Omega(K^\times)$ , defined by

$$Z_\infty(\omega; 1/\rho) = \int_{X_K} \Phi_\infty(y) \omega((1/\rho)(y)) |\sigma^* dx|_K$$

for  $\omega \in \Omega(K^\times)$ , which belongs to the class  $\mathcal{Z}$ . In particular,  $F^\#$  has the termwise differentiable and uniform asymptotic expansion

$$(4.11) \quad F^\#(t) \approx \sum_{\lambda \in \Lambda} \sum_{m=1}^{m_\lambda} b_{\lambda, m} \left( \frac{t}{|t|} \right) |t|_K^{\lambda-1} (\ln |t|_K)^{m-1} \text{ as } |t|_K \rightarrow 0,$$

where the  $b_{\lambda,m}$  are smooth functions on  $\{u \in K^\times \mid |u|_K = 1\}$ , and where  $\{-\lambda \mid \lambda \in \Lambda\}$  is the set of poles of  $Z_\infty(\omega; 1/\rho)$  (for all  $\omega$ ). Note that if  $-\lambda$  is a (necessarily negative) pole of  $Z_\infty(\omega; 1/\rho)$ , it corresponds to the (positive) pole  $\lambda$  of  $Z_\infty(\omega^{-1}; 1/\rho)$ , and thus to the positive pole  $\lambda$  of  $Z_\Phi(\omega; f/g)$ .

Next, we consider the fibre integral

$$F_\infty(u) := \int_{\rho(y)=u} \Phi_\infty(y) \left| \frac{\sigma^* dx}{d\rho} \right|_K,$$

for  $u \in K^\times$ . Clearly,

$$(4.12) \quad E_\infty(z) = \int_{K^\times} \Psi(z \cdot u) F_\infty(u) |du|_K,$$

and hence it is the Fourier transform of  $F_\infty(u)$ . Also, we have, with the change of coordinates  $t = 1/u$ , that  $F_\infty(u) = |u|_K^{-2} F^\#(1/u)$ . Indeed, since  $\sigma^* dx$  is equal to both

$$\frac{\sigma^* dx}{d\rho} \wedge d\rho \quad \text{and} \quad \frac{\sigma^* dx}{d(1/\rho)} \wedge d(1/\rho) = -\frac{1}{\rho^2} \frac{\sigma^* dx}{d(1/\rho)} \wedge d\rho,$$

and the restriction of the Gel'fand-Leray form to any fibre  $\rho = u$  is unique, we have

$$|-\rho^2|_K \left| \frac{\sigma^* dx}{d\rho} \right|_K = \left| \frac{\sigma^* dx}{d(1/\rho)} \right|_K.$$

Since  $F^\#$  is a Schwartz function at infinity, we can extend  $F_\infty$  to a function on  $K$  by declaring  $F_\infty(0) := 0$ , satisfying  $\frac{d^k}{du^k} F_\infty(u) \Big|_{u=0} = 0$  for any  $k$ . So

- (i)  $F_\infty \in C^\infty(K)$ ;
- (ii) we have the termwise differentiable and uniform asymptotic expansion

$$(4.13) \quad \begin{aligned} F_\infty(u) &\approx |u|_K^{-2} \sum_{\lambda} \sum_{m=1}^{m_\lambda} b_{\lambda,m} \left( \left( \frac{u}{|u|} \right)^{-1} \right) |u|_K^{-\lambda+1} (\ln |u^{-1}|_K)^{m-1} \\ &\approx \sum_{\lambda} \sum_{m=1}^{m_\lambda} (-1)^{m-1} b_{\lambda,m} \left( \left( \frac{u}{|u|} \right)^{-1} \right) |u|_K^{-\lambda-1} (\ln |u|_K)^{m-1} \end{aligned}$$

as  $|u|_K \rightarrow +\infty$ .

Hence  $F_\infty$  belongs to the space  $\mathcal{G}^*$  of Theorem 4.5. Then, by (4.12), Theorem 4.5 and Proposition 4.6, we obtain

- (i)  $E_\infty \in C^\infty(K^\times)$ ;
- (ii)  $E_\infty$  behaves like a Schwartz function of  $z$  as  $|z|_K$  tends to infinity;
- (iii) we have the asymptotic expansion

$$(4.14) \quad E_\infty(z) \approx \sum_{\lambda} \sum_{m=1}^{m_\lambda + \delta_\lambda} A_{\lambda,m} \left( \frac{z}{|z|} \right) |z|_K^\lambda (\ln |z|_K)^{m-1} \quad \text{as } |z|_K \rightarrow 0,$$

where  $\lambda$  runs over  $\Lambda \cup d_K \mathbb{Z}_{\geq 0}$ . More precisely, we have the following. By Proposition 4.6(3),  $w_\chi(\lambda + 1)$  can only be zero if  $\lambda \in d_K \mathbb{Z}_{\geq 0}$ . Hence, if  $\lambda \notin d_K \mathbb{Z}_{\geq 0}$ , or if  $\lambda \in d_K \mathbb{Z}_{\geq 0}$  and  $m \leq m_\lambda$ , the coefficients  $A_{\lambda,m}$  in (4.14) are explicitly determined by the coefficients  $b_{\lambda,m}$  in (4.13). But for  $\lambda \in d_K \mathbb{Z}_{\geq 0}$ , the coefficients  $A_{\lambda, m_\lambda + 1}$  are not determined by (4.13), and can be nonzero.

In particular,  $|z|_K^0$  can appear with nonzero coefficient  $A_{0,1}$  in (4.14). We claim that this coefficient must be a constant. Indeed, again by Proposition 4.6(3),  $w_{\chi_l}(1) = 0$  if and only if  $1 = 1 + \frac{1}{[K:\mathbb{R}]}(l + 2k)$  for some  $k \in \mathbb{Z}_{\geq 0}$ . This can

only occur when  $l = 0$ . Hence, writing  $A_{0,1,\chi_l}$  for the Fourier coefficients of  $A_{0,1}$ , i.e.,

$$A_{0,1} = \sum_l A_{0,1,\chi_l} \chi_l,$$

we obtain from (4.5) that  $A_{0,1,\chi_l} = 0$  for  $l \neq 0$ . Since  $\chi_0$  is the trivial character, we conclude that  $A_{0,1}$  is the constant function  $A_{0,1,\chi_0}$ .

This is consistent with the fact that we should have a constant in the expansion of  $E_\infty(z)$ : from its definition it should be  $E_\infty(0) = \int_{X_K \setminus \sigma^{-1}D_K} (\Phi_\infty)(y) |\sigma^* dx|_K$ . Note that, in the formulation of the theorem, we put this constant on the left hand side; for this reason we must consider only  $\lambda > 0$  in the asymptotic expansion.  $\square$

**Remark 4.11.** Related to the last part of the proof above: according to [29, II Theorem 3.1],  $E_\infty(0)$  exists if and only if  $F_\infty \in L^1(K)$ . This last assertion is indeed true, since the powers of  $|u|_K$  in the expansion (4.13) are (strictly) smaller than  $-1$ .

The treatment of oscillatory integrals over  $p$ -fields is completely similar.

**Proposition 4.12.** *Let  $K$  be a  $p$ -field. We denote by  $m_\lambda$  the order of a pole  $\lambda$  of  $Z_\Phi(\omega; f/g)$ .*

(1) *Then  $E_0(z)$  has an expansion of the form*

$$(4.15) \quad E_0(z) = \sum_{\lambda < 0} \sum_{m=1}^{m_\lambda} A_{\lambda,m}(ac\,z) |z|_K^\lambda (\ln |z|_K)^{m-1}$$

*for sufficiently large  $|z|_K$ , where  $\lambda$  runs through all the poles mod  $2\pi i / \ln q$  with negative real part of  $Z_\Phi(\omega; f/g)$  (for all  $\omega$ ), and each  $A_{\lambda,m}$  is a locally constant function on  $R_K^\times$ .*

(2) *Writing  $\omega = \omega_s \chi(ac)$ , we have that  $A_{\lambda,m_\lambda} = 0$  if  $\chi = 1$  and  $\lambda = -1$  mod  $2\pi i / \ln q$ .*

*Proof.* This can be shown using the same arguments as in the proof of Proposition 4.9. The refinement now follows from [29, I Theorem 5.3] and Proposition 4.6.  $\square$

**Proposition 4.13.** *Let  $K$  be a  $p$ -field. We denote by  $m_\lambda$  the order of a pole  $\lambda$  of  $Z_\Phi(\omega; f/g)$ . Then  $E_\infty(z)$  has an expansion of the form*

$$(4.16) \quad E_\infty(z) - C = \sum_{\lambda > 0} \sum_{m=1}^{m_\lambda} A_{\lambda,m}(ac\,z) |z|_K^\lambda (\ln |z|_K)^{m-1}$$

*for sufficiently small  $|z|_K$ , where  $C = E_\infty(0)$  is a constant,  $\lambda$  runs through all the poles mod  $2\pi i / \ln q$  with positive real part of  $Z_\Phi(\omega; f/g)$  (for all  $\omega$ ), and each  $A_{\lambda,m}$  is a locally constant function on  $R_K^\times$ .*

*Proof.* This can be shown using the same arguments as in the proof of Proposition 4.10. For  $p$ -fields however, Proposition 4.6(3) is ‘easier’, saying that  $w_\chi(\lambda + 1) = 0$  only if  $\chi = 1$  and  $\lambda = 0$  mod  $2\pi i / \ln q$ , yielding as only ‘extra term’ the constant  $C = E_\infty(0)$ .  $\square$



## 5. EXPANSIONS FOR OSCILLATORY INTEGRALS

**5.1. Expansion for  $E_\Phi(z; f/g)$  as  $z$  approaches zero.** We start with an obvious lemma, and show explicitly the result for  $\mathbb{R}$ -fields. The proof for  $p$ -fields is analogous.

**Lemma 5.1.** *With the notation of Subsection 4.3, we have that*

$$\lim_{z \rightarrow 0} E_1(z) = \lim_{z \rightarrow 0} E_3(z) = \int_{K^n} \Theta(y) \left( \prod_{i=1}^n |y_i|_K^{v_i-1} \right) |dy|_K.$$

*Proof.* This follows from the Lebesgue Dominated Convergence Theorem.  $\square$

**Theorem 5.2.** *Let  $K$  be an  $\mathbb{R}$ -field; we put  $d_K := 1$  if  $K = \mathbb{R}$  and  $d_K := 1/2$  if  $K = \mathbb{C}$ . We take  $\frac{f}{g} : U \rightarrow K$  and  $\Phi$  as in Section 2. If  $Z_\Phi(\omega; f/g)$  has a positive pole for some  $\omega$ , then*

$$E_\Phi(z; f/g) - \int_{K^n} \Phi(x) |dx|_K \approx \sum_{\gamma > 0} \sum_{m=1}^{m_\gamma + \delta_\gamma} e_{\gamma, m} \left( \frac{z}{|z|} \right) |z|_K^\gamma (\ln |z|_K)^{m-1} \text{ as } |z|_K \rightarrow 0,$$

where  $\gamma$  runs through  $d_K \mathbb{Z}_{>0}$  and all the positive poles of  $Z_\Phi(\omega; f/g)$ , for all  $\omega \in \Omega(K^\times)$ , that are not in  $d_K \mathbb{Z}_{>0}$ , with  $m_\gamma$  the order of  $\gamma$ , and with  $\delta_\gamma = 0$  if  $\gamma \notin d_K \mathbb{Z}_{>0}$  and  $\delta_\gamma = 1$  if  $\gamma \in d_K \mathbb{Z}_{>0}$ . Finally, each  $e_{\gamma, m}$  is a smooth function on  $\{u \in K^\times \mid |u|_K = 1\}$ .

*Proof.* By the definitions in Subsections 4.3 and 4.4, we have that  $E_\Phi(z; f/g)$  is the sum of  $E_\infty(z)$  and a linear combination of elementary integrals of types  $E_1(z)$  and  $E_3(z)$ . By Lemma 5.1, the contributions of integrals of type  $E_1(z)$  and  $E_3(z)$  are constants. The asymptotic expansion of  $E_\infty(z)$  was calculated in Proposition 4.10. Finally, the sum of all constants that appear in the calculations equals  $\int_{K^n} \Phi(x) |dx|_K$ , by the Lebesgue Dominated Convergence Theorem.  $\square$

**Theorem 5.3.** *Let  $K$  be a  $p$ -field. We take  $\frac{f}{g} : U \rightarrow K$  and  $\Phi$  as in Section 2. If  $Z_\Phi(\omega; f/g)$  has a pole with positive real part for some  $\omega$ , then*

$$E_\Phi(z; f/g) - \int_{K^n} \Phi(x) |dx|_K = \sum_{\gamma > 0} \sum_{m=1}^{m_\gamma} e_{\gamma, m} (ac z) |z|_K^\gamma (\ln |z|_K)^{m-1} \text{ for sufficiently small } |z|_K,$$

where  $\gamma$  runs through all the poles mod  $2\pi i / \ln q$  with positive real part of  $Z_\Phi(\omega; f/g)$ , for all  $\omega \in \Omega(K^\times)$ , and with  $m_\gamma$  the order of  $\gamma$ . Finally, each  $e_{\gamma, m}$  is a locally constant function on  $R_K^\times$ .

**5.2. Expansion for  $E_\Phi(z; f/g)$  as  $z$  approaches infinity.** We start with two useful preliminary facts.

**Lemma 5.4.** *Let  $K$  be an  $\mathbb{R}$ -field or a  $p$ -field. Let  $h : V \rightarrow K$  be a non-constant  $K$ -analytic function on an open set  $V$  in  $K^n$ , and  $D$  a compact subset of  $V$ . Then*

the restriction of  $h$  to a small neighborhood of  $D$  has only finitely many critical values.

*Proof.* This is well known to the experts. For  $K = \mathbb{R}$ , it follows from an appropriate version of the curve selection lemma, see e.g. [38, Corollary 1.6.1]. We sketch an alternative argument that also works for  $p$ -fields  $K$ . Consider the restriction of  $h$  to a small neighborhood of  $D$ . Because  $D$  is compact, the image of the critical set of this restriction is globally subanalytic in  $K$  (we refer to [16],[40],[10] for  $p$ -fields). Since, by Sard's theorem, this image is also of measure zero, it can only consist of a finite set of points.

For  $K = \mathbb{C}$ , the statement follows from the real case by considering  $h$  as a map from  $V \subset \mathbb{R}^{2n}$  to  $\mathbb{R}^2$  and by the Cauchy-Riemann equations.  $\square$

As before, let  $\Theta$  be the characteristic function of a small polydisc around the origin, or a smooth function supported in the polydisc

$$\{y \in K^n \mid |y_j|_K < 1 \text{ for } j = 1, \dots, n\},$$

according as  $K$  is a  $p$ -field or an  $\mathbb{R}$ -field. We consider a special case of the integral  $E_3$  of (4.8), namely

$$\begin{aligned} E'_3(z) &:= \int_{K^n} \Theta(y) \Psi(z \cdot (1 + y_1)) \left( \prod_{i=2}^n |y_i|_K^{v_i-1} \right) |dy|_K \\ (5.1) \quad &= \Psi(z) \int_{K^n} \Theta(y) \Psi(z \cdot y_1) \left( \prod_{i=2}^n |y_i|_K^{v_i-1} \right) |dy|_K. \end{aligned}$$

Note that  $y_1$  does not occur in the product in the integrand.

**Lemma 5.5.** (1) Let  $K$  be an  $\mathbb{R}$ -field. Then  $E'_3(z) = O(|z|_K^{-k})$  as  $|z|_K \rightarrow \infty$ , for any positive number  $k$ .

(2) Let  $K$  be a  $p$ -field. Then  $E'_3(z) = 0$  for sufficiently large  $|z|_K$ .

*Note.* This can be considered as a special case of (the proof of) Proposition 4.9(ii) and Proposition 4.12(ii), but we think that it is appropriate to mention an explicit elementary proof.

*Proof.* (1) If  $K = \mathbb{R}$ , we can rewrite  $E'_3(z)$  by a standard computation as

$$e^{2\pi\sqrt{-1}z} \int_{\mathbb{R}} \Omega(y_1) e^{2\pi\sqrt{-1}zy_1} |dy_1|_{\mathbb{R}},$$

where  $\Omega(y_1)$  is a smooth function supported in  $\{y_1 \in \mathbb{R} \mid |y_1|_{\mathbb{R}} < 1\}$  (by Lebesgue's Dominated Convergence Theorem). The result follows by using repeated integration by parts.

If  $K = \mathbb{C}$ , we take  $y_1 = y_{11} + \sqrt{-1}y_{12}$  with  $y_{11}, y_{12} \in \mathbb{R}$ , and  $z = z_1 + \sqrt{-1}z_2 = |z|_{\mathbb{C}}(\tilde{z}_1 + \sqrt{-1}\tilde{z}_2)$ , with  $z_1, z_2 \in \mathbb{R}$ . Then  $E'_3(z)$  can be rewritten as

$$\begin{aligned} &e^{4\pi\sqrt{-1}z_1} \int_{\mathbb{R}^2} \Omega(y_{11}, y_{12}) e^{4\pi\sqrt{-1}(y_{11}z_1 - y_{12}z_2)} |dy_{11}dy_{12}|_{\mathbb{R}} \\ (5.2) \quad &= e^{4\pi\sqrt{-1}z_1} \int_{\mathbb{R}^2} \Omega(y_{11}, y_{12}) e^{4\pi\sqrt{-1}|z|_{\mathbb{C}}(y_{11}\tilde{z}_1 - y_{12}\tilde{z}_2)} |dy_{11}dy_{12}|_{\mathbb{R}}, \end{aligned}$$

where  $\Omega$  is a smooth function with support in  $\{(y_{11}, y_{12}) \in \mathbb{R}^2 \mid y_{11}^2 + y_{12}^2 < 1\}$ . Since we are interested in the behavior of (5.2) for  $|z|_{\mathbb{C}}$  big enough, we may assume that  $z_1$  or  $z_2$  is big enough; we consider for example the case that  $z_1$  is big enough. Fixing  $\tilde{z}_1$  and  $\tilde{z}_2$ , we perform the change of variables  $y_{11} = \frac{u_1}{\tilde{z}_1} + \frac{u_2 \tilde{z}_2}{\tilde{z}_1}$ ,  $y_{12} = u_2$ , reducing the asymptotics of (5.2) to the asymptotics of the integrals considered in the case  $K = \mathbb{R}$ .

(2) Say  $\Theta$  is the characteristic function of  $(P^e R_K)^n$ . Then

$$E'_3(z) = C \Psi(z) \int_{P^e R_K} \Psi(z \cdot y_1) |dy_1|_K,$$

with  $C$  a constant, and this integral vanishes when  $|z|_K > q^{-e}$ .  $\square$

**Remark 5.6.** In Lemma 5.5, it is crucial that the integrand in (5.1) does not contain a factor  $|y_1|_K^{v_1-1}$  with  $v_1 > 1$ . Indeed, for example in the  $p$ -field case we have that an integral like

$$\int_{\mathbb{Z}_p} \Psi(z(1+y)) |y|_{\mathbb{Q}_p}^{v-1} |dy|_{\mathbb{Q}_p}$$

does not vanish for sufficiently large  $|z|_{\mathbb{Q}_p}$ . (A straightforward computation shows that it equals

$$\Psi(z) \frac{1-p^{v-1}}{1-p^{-v}} |z|_{\mathbb{Q}_p}^{-v}$$

when  $|z|_{\mathbb{Q}_p} \geq p$ .) Similarly, when  $K = \mathbb{R}$ , consider the integral

$$\int_{\mathbb{R}} \Phi(y) \Psi(z(1+y)) |y|_{\mathbb{R}}^{v-1} |dy|_{\mathbb{R}}$$

with  $v$  even,  $\Phi$  positive and  $\Phi(0) > 0$ . Then in the asymptotic expansion of this integral as  $|z|_{\mathbb{R}} \rightarrow \infty$ , we have that the term  $\Psi(z) |z|_{\mathbb{R}}^{-v}$  appears with non-zero coefficient, see e.g. [2, p. 183]. In particular, the assertion of Lemma 5.5(1) cannot be true.

For an analytic function  $f$ , the only relevant expansion of  $E_{\Phi}(z; f)$  is as  $|z|_K \rightarrow \infty$ . It depends on all critical points of  $f$ , but typically one assumes that

$$(5.3) \quad C_f \cap \text{supp } \Phi \subset f^{-1}(0).$$

This assumption is not really restrictive, because, by Lemma 5.4,  $f$  has only finitely many critical values on the support of  $\Phi$ . Hence, one can reduce to the situation (5.3) by partitioning the original support of  $\Phi$  and applying a translation. Note however that, when  $c \in K^{\times}$  is a critical value of  $f$  and  $C_f \cap \text{supp } \Phi \subset f^{-1}(c)$ , we have that

$$E_{\Phi}(z; f) = E_{\Phi}(z; c + (f - c)) = \Psi(c \cdot z) E_{\Phi}(z; f - c).$$

Therefore, in expansions like in Proposition 4.9, all terms will contain an additional factor  $\Psi(c \cdot z)$ .

The explicit expansion of  $E_{\Phi}(z; f)$  as  $|z|_K \rightarrow \infty$ , without any assumption on  $C_f \cap \text{supp } \Phi$ , and hence involving factors  $\Psi(c \cdot z)$  for all critical values  $c$  of  $f$ , was treated for  $p$ -fields in [21, Proposition 2.7] (when  $f$  is a polynomial).

Now for our object of study  $E_{\Phi}(z; f/g)$ , one *cannot* reduce the problem to a case like (5.3) by pulling back the integral  $E_{\Phi}(z; f/g)$  to the manifold  $X_K$  and by restricting the support of  $\Phi$ . Indeed, suppose that  $f$  and  $g$  have at least one

common zero in  $K^n$ , say  $P$  (this is of course the interesting situation). Fixing an adapted embedded resolution  $\sigma$  of  $D_K$ , it turns out that, as soon as  $P \in \text{supp } \Phi$ , one must take into account factors  $\Psi(c \cdot z)$ , where  $c$  is a critical value of  $\rho = (f/g) \circ \sigma$  or another ‘special’ value of  $\rho$ , defined below.

We keep using the notation of Subsection 4.3. In particular we consider around each  $b \in X_K$  local coordinates  $y$  in a small enough polydisc  $B$ , such that  $|\sigma^* dx|_K$  is locally of the form  $\prod_{i=1}^n |y_i|_K^{v_i-1} |dy|_K$ , and  $\rho$  is locally in  $y$  a monomial, the inverse of a monomial, or invertible, according as  $\rho(b)$  is 0,  $\infty$ , or in  $K^\times$ .

**Definition 5.7.** We take  $\frac{f}{g} : U \rightarrow K$  and  $\Phi$  as in Section 2. We fix an adapted embedded resolution  $\sigma$  of  $D_K$ , for which we use the notation of Subsection 4.3.

For a point  $b \in \sigma^{-1} D_K$  that is not a critical point of  $\rho$  and for which  $c = \rho(b) \in K^\times$ , we have in local coordinates  $y$  that  $\rho$  is of the form  $c + \ell(y) + h(y)$ , where  $\ell(y)$  is a non-zero linear form and  $h(y)$  consists of higher order terms. We define  $\mathcal{S} \subset X_K$  as the set of *special points* of  $\rho$ , being the union of the critical points of  $\rho$  and the non-critical points  $b$  as above for which  $\ell(y)$  is linearly dependent on the  $y_i$  with  $v_i > 1$ .

**Example 5.8.** With notation as before, we take  $U = K^2$  and  $\Phi$  such that  $\Phi(0) \neq 0$ . Let

$$\begin{aligned} (1) \quad & f = x^2 - y^2 \quad \text{and} \quad g = x^2, \quad \text{or} \\ (2) \quad & f = x^2 + x^3 - y^2 \quad \text{and} \quad g = x^2. \end{aligned}$$

In both cases we obtain an adapted embedded resolution  $\sigma : X_K \rightarrow K^2$  simply by blowing up at the origin of  $K^2$ . Consider the chart  $V$  of  $X_K$  with coordinates  $x_1, y_1$  given by  $x = x_1, y = x_1 y_1$ . On  $V$  the exceptional curve  $E_1$  is given by  $x_1 = 0$  and  $\rho = (f/g) \circ \sigma$  is given by

$$(1) \quad \rho = 1 - y_1^2 \quad \text{or} \quad (2) \quad \rho = 1 + x_1 - y_1^2,$$

respectively. The point  $(x_1, y_1) = (0, 0)$  is a critical point of  $\rho$  in case (1) and a non-critical special point of  $\rho$  in case (2). One easily verifies that in both cases there are no other special points.

Note that in the classical theory of Igusa, studying expansions of polynomial/analytic functions  $f$ , a non-critical special point cannot occur. Indeed, with the notation of Definition 5.7, then all  $v_i = 1$  for a point  $b \in X_K$  satisfying  $\rho(b) \in K^\times$ .

**Lemma 5.9.** *The set  $\rho(\mathcal{S} \cap \text{supp}(\sigma^* \Phi))$  is finite.*

*Proof.* In order to simplify notation, we assume for each point  $b \in \text{supp}(\sigma^* \Phi)$  that exactly  $y_{k+1}, \dots, y_n$  satisfy  $v_i > 1$ , where  $k \geq 0$ . (This can be achieved after a permutation of the coordinates  $y$  in  $B$ .) We consider in  $B$  the subset

$$\mathcal{S}_b := \{a \in B \cap \mathcal{S} \mid a \text{ has local coordinates } (a_1, a_2, \dots, a_k, 0, \dots, 0)\}.$$

The union of all these  $\mathcal{S}_b$  contains  $\mathcal{S} \cap \text{supp}(\sigma^* \Phi)$ . Hence, by compactness of  $\text{supp}(\sigma^* \Phi)$ , it is enough to show that each  $\rho(\mathcal{S}_b)$  is finite.

Note that there are only finitely many points  $b \in X_K$  for which all  $v_i > 1$ , because they correspond to the (finitely many) intersection points of exactly  $n$  of the components  $E_j, j \in T$ , for which  $E_j \cap \text{supp}(\sigma^* \Phi) \neq \emptyset$ . Therefore, we may take  $k \geq 1$  above.

Assume first that  $\rho(b) \in K^\times$ , and write  $\rho$  in the local coordinates  $y$  as

$$w(y_1, \dots, y_k) + w'(y_1, \dots, y_n),$$

where  $w$  depends only on  $y_1, \dots, y_k$  and  $w'$  belongs to the ideal generated by  $y_{k+1}, \dots, y_n$ . (In particular the constant term of  $w$  is precisely  $\rho(b)$ .) By definition of  $\mathcal{S}$ , the points  $a = (a_1, a_2, \dots, a_k, 0, \dots, 0)$  in  $\mathcal{S}_b$  satisfy

$$0 = \frac{\partial \rho}{\partial y_i}(a) = \frac{\partial w}{\partial y_i}(a_1, \dots, a_k)$$

for all  $i = 1, \dots, k$ . Hence, these are critical points of the function  $w$ , and then, by Lemma 5.4, we conclude that  $w(\mathcal{S}_b) = \rho(\mathcal{S}_b)$  is finite.

When  $\rho(b) = 0$ , the function  $\rho$  is monomial in the local coordinates  $y$ , and hence  $\rho(B \cap \mathcal{S})$  is  $\{0\}$  or empty. Similarly, when  $\rho(b) = \infty$ , we have that  $\rho(B \cap \mathcal{S})$  is  $\{\infty\}$  or empty.  $\square$

**Definition 5.10.** We denote  $\mathcal{V} := \rho(\mathcal{S} \cap \text{supp}(\sigma^* \Phi)) \setminus \{\infty\}$ , the set of *special values*.

Here we discard  $\infty$  in order to simplify the notation for the summation set in the theorems below. In Example 5.8, we have twice  $\mathcal{V} = \{1\}$ .

**Theorem 5.11.** Let  $K$  be an  $\mathbb{R}$ -field. We take  $\frac{f}{g} : U \rightarrow K$  and  $\Phi$  as in Section 2. Let  $\mathcal{V}$  be the set of special values as in Definition 5.10.

(1) If  $Z_\Phi(\omega; f/g - c)$  has a negative pole for some  $c \in \mathcal{V}$  and some  $\omega$ , then

$$E_\Phi(z; f/g) \approx \sum_{c \in \mathcal{V}} \sum_{\gamma_c < 0} \sum_{m=1}^{m_{\gamma_c}} e_{\gamma_c, m, c} \left( \frac{z}{|z|} \right) \Psi(c \cdot z) |z|_K^{\gamma_c} (\ln |z|_K)^{m-1} \text{ as } |z|_K \rightarrow \infty,$$

where  $\gamma_c$  runs through all the negative poles of  $Z_\Phi(\omega; f/g - c)$ , for all  $\omega \in \Omega(K^\times)$ , and with  $m_{\gamma_c}$  the order of  $\gamma_c$ . Finally, each  $e_{\gamma_c, m, c}$  is a smooth function on  $\{u \in K^\times \mid |u|_K = 1\}$ .

(2) Writing  $\omega = \omega_s \chi_l(ac)$ , we have that  $e_{\gamma_c, m_{\gamma_c}, c} = 0$  if  $|\gamma_c| \in 1 + \frac{1}{[K:\mathbb{R}]}(|l| + 2\mathbb{Z}_{\geq 0})$ .

*Proof.* We construct an embedded resolution  $\tilde{\sigma} : X'_K \rightarrow X_K$  of

$$D'_K := \sigma^{-1}(D_K) \cup (\cup_{c \in \mathcal{V} \setminus \{0\}} \rho^{-1}\{c\}),$$

where we assume that  $\tilde{\sigma}$  is a  $K$ -analytic isomorphism outside the inverse image of  $D'_K$ . Since  $c \neq 0, \infty$  above, we have in particular that  $\tilde{\sigma}$  is an isomorphism on sufficiently small neighborhoods of  $\rho^{-1}\{0\}$  and  $\rho^{-1}\{\infty\}$ .

Instead of working with  $\sigma$  and  $\rho$ , we now work with the compositions  $\sigma' = \sigma \circ \tilde{\sigma} : X'_K \rightarrow U$  and  $\rho' = \rho \circ \tilde{\sigma} : X'_K \rightarrow \mathbb{P}^1$ . We also use now  $y = (y_1, \dots, y_n)$  as local coordinates on  $X'_K$ .

Again, we study

$$E_\Phi(z; f/g) = \int_{X'_K \setminus (\sigma')^{-1} D_K} ((\sigma')^* \Phi)(y) \Psi(z \cdot \rho'(y)) |(\sigma')^* dx|_K$$

via local contributions, similarly as in Subsection 4.4, but now fixing a more refined appropriate decomposition of  $(\sigma')^* \Phi$ .

We cover the (compact) support of  $(\sigma')^* \Phi$  by finitely many polydiscs  $B_j, j \in J$ , as before, making sure that the (compact) fibres  $(\rho')^{-1}\{c\} \cap \text{supp}((\sigma')^* \Phi)$ , for all  $c \in \mathcal{V} \cup \{0, \infty\}$ , are completely covered by certain  $B_j, j \in J_c \subset J$ , with centre  $b_j$  mapped by  $\rho$  to  $c$ . (Note that possibly  $0 \notin \mathcal{V}$ , as in Example 5.8.) We define  $\Phi_c$ , for

$c \in \mathcal{V} \cup \{0, \infty\}$ , as the restriction of  $(\sigma')^* \Phi$  to  $\cup_{j \in J_c} B_j$ , modified by the partition of the unity that induces the functions  $\Theta$  of Subsection 4.3 on the local charts  $B_j$ .

Besides the contributions  $E_0(z)$  and  $E_\infty(z)$  as in Subsection 4.4, we now define also for  $c \in \mathcal{V} \setminus \{0\}$  the contribution

$$E_c(z) = \int_{X'_K \setminus (\sigma')^{-1} D_K} (\Phi_c)(y) \Psi(z \cdot \rho'(y)) |(\sigma')^* dx|_K.$$

Then we have that  $E_\Phi(z; f/g)$  is the sum of  $E_\infty(z)$ ,  $E_0(z)$ , the  $E_c(z)$  with  $c \in \mathcal{V} \setminus \{0\}$ , and of a linear combination of elementary integrals of type  $E_3(z)$  as in (4.8) for which  $c = \rho'(b) \notin \mathcal{V}$ . We first consider this last type of contributions. We rewrite

$$E_3(z) = \Psi(z \cdot c) \int_{K^n} \Theta(y) \Psi(z \cdot (u(y) - c)) \left( \prod_{i=1}^n |y_i|_K^{v_i-1} \right) |dy|_K.$$

Then, maybe after restricting the support of  $\Theta$ , the integral above can be rewritten in the form (5.1) by a change of coordinates. By Lemma 5.5, there is no contribution to the asymptotic expansion as  $|z|_K \rightarrow \infty$ .

Next, we saw in the proof of Proposition 4.10 that  $E_\infty(z)$  behaves like a Schwartz function as  $|z|_K \rightarrow \infty$ , and hence also does not contribute to the asymptotic expansion of  $E_\Phi(z; f/g)$ . The expansion of  $E_0(z)$  was calculated in Proposition 4.9. Finally, we apply (the proof of) Proposition 4.9 to obtain for  $c \in \mathcal{V} \setminus \{0\}$  the similar expansion

$$\Psi(c \cdot z) \sum_{\gamma_c < 0} \sum_{m=1}^{m_{\gamma_c}} e_{\gamma_c, m, c} \left( \frac{z}{|z|} \right) |z|_K^{\gamma_c} (\ln |z|_K)^{m-1} \text{ as } |z|_K \rightarrow \infty$$

of  $E_c(z)$ , together with the vanishing property in (2). All together, this yields the expansion in the statement, summing over all  $c \in \mathcal{V}$ . Note that indeed, in the special case where  $0 \notin \mathcal{V}$ , also  $E_0(z)$  behaves like a Schwartz function as  $|z|_K \rightarrow \infty$  and does not contribute to the total asymptotic expansion (by Proposition 4.9(ii) or Lemma 5.5).  $\square$

The following similar result in the  $p$ -field case is proven analogously, now invoking Propositions 4.12 and 4.13 instead of their real versions.

**Theorem 5.12.** *Let  $K$  be a  $p$ -field. We take  $\frac{f}{g} : U \rightarrow K$  and  $\Phi$  as in Section 2. Let  $\mathcal{V}$  be the set of special values as in Definition 5.10.*

(1) *If  $Z_\Phi(\omega; f/g - c)$  has a pole with negative real part for some  $c$  and some  $\omega$ , then*

$$E_\Phi(z; f/g) = \sum_{c \in \mathcal{V}} \sum_{\gamma_c < 0} \sum_{m=1}^{m_{\gamma_c}} e_{\gamma_c, m, c} (ac z) \Psi(c \cdot z) |z|_K^{\gamma_c} (\ln |z|_K)^{m-1}$$

*for sufficiently large  $|z|_K$ , where  $\gamma_c$  runs through all the poles mod  $2\pi\sqrt{-1}/\ln q$  with negative real part of  $Z_\Phi(\omega; f/g - c)$ , for all  $\omega \in \Omega(K^\times)$ , and with  $m_{\gamma_c}$  the order of  $\gamma_c$ . Finally, each  $e_{\gamma_c, m, c}$  is a locally constant function on  $R_K^\times$ .*

(2) *Writing  $\omega = \omega_s \chi(ac)$ , we have that  $e_{\gamma_c, m_{\gamma_c}, c} = 0$  if  $\chi = 1$  and  $\gamma_c = -1 \bmod 2\pi i / \ln q$ .*

**Remark 5.13.** The form of our expansions in Theorems 5.3 and 5.12 is consistent with the results in [11] and [13], where a similar form of such expansions is derived

for  $p$  big enough, in a more general setting, but without information on the powers of  $|z|_K$  and  $\ln|z|_K$ .

### 5.3. Examples.

**Example 5.14.** We continue the study of case (1) of Example 5.8 when  $K$  is a  $p$ -field. In that case the map  $\sigma$  is also an embedded resolution of  $\rho^{-1}\{1\}$ . The exceptional curve  $E_1$  has datum  $(N_1, v_1) = (0, 2)$ .

(1) The fibre  $\rho^{-1}\{0\}$  is the strict transform of  $f^{-1}\{0\}$  and consists of two (dis-joint) components with datum  $(1, 1)$ . Hence the only possible negative real part of a pole of  $Z_\Phi(\omega; f/g)$  is  $-1$ . Moreover, by Remark 3.4, such a pole occurs only if  $\chi$  is trivial, and it must be of the form  $-1 + \frac{2\pi\sqrt{-1}}{\ln q}k, k \in \mathbb{Z}$ .

The fibre  $\rho^{-1}\{1\}$  is given (as a set) by  $y_1 = 0$  in the chart  $V$  and it has datum  $(2, 1)$ , with respect to the function  $\rho - 1$ . Then the only possible negative real part of a pole of  $Z_\Phi(\omega; f/g - 1)$  is  $-\frac{1}{2}$ . By Theorem 3.2 and Remark 3.4, it can only occur if the order of  $\chi$  is 1 or 2, and it must be of the form  $-\frac{1}{2} + \frac{\pi\sqrt{-1}}{\ln q}k, k \in \mathbb{Z}$ . (Clearly all poles are of order 1.)

Hence Theorem 5.12 predicts the following possible terms in the expansion of  $E_\Phi(z; f/g)$  for sufficiently large  $|z|_K$ :  $\Psi(z)|z|_K^{-1/2}$  and  $\Psi(z)|z|_K^{-1/2}(-1)^{\text{ord}(z)}$ . Note that there should be no term  $|z|_K^{-1}$  because of part (2) of that theorem.

As an illustration, we consider the concrete case  $K = \mathbb{Q}_p$  with  $p \neq 2$ , and  $\Phi$  the characteristic function of  $\mathbb{Z}_p^2$ . Then, using Theorem 3.15 or with elementary calculations, one can verify that

$$(5.4) \quad Z_\Phi(\omega; f/g) = \frac{p^{1+s} + p^2(p-2)p^{-s} + p^{2-2s} - 2p + 1}{(p+1)(p^{1+s} - 1)(p^{1-2s} - 1)}$$

and

$$Z_\Phi(\omega; f/g - 1) = \frac{(p-1)^2}{(p^{1+2s} - 1)(p^{1-2s} - 1)}.$$

And with some more effort, one can compute the following explicit expression for  $E_\Phi(z; f/g)$  in this case:

$$E_\Phi(z; f/g) = \frac{p}{p+1} \Psi(z) \eta_p(-z) |z|_{\mathbb{Q}_p}^{-1/2}$$

for sufficiently large  $|z|_{\mathbb{Q}_p}$ . Here

$$\eta_p(a) = \begin{cases} 1 & \text{if } \text{ord}(a) \text{ is even} \\ \left(\frac{a_0}{p}\right) & \text{if } \text{ord}(a) \text{ is odd and } p \equiv 1 \pmod{4} \\ \sqrt{-1} \left(\frac{a_0}{p}\right) & \text{if } \text{ord}(a) \text{ is odd and } p \equiv 3 \pmod{4}, \end{cases}$$

where  $a = p^{\text{ord}(a)}(a_0 + a_1p + a_2p^2 + \dots)$  and the  $a_i \in \{0, \dots, p-1\}$ . At first sight, this may seem different from the description above, but  $\eta_p(-z)|z|_{\mathbb{Q}_p}^{-1/2}$  can be written as a combination of  $|z|_{\mathbb{Q}_p}^{-1/2}$  and  $|z|_{\mathbb{Q}_p}^{-1/2}(-1)^{\text{ord}(z)}$ .

(2) The fibre  $\rho^{-1}\{\infty\}$  is the strict transform of  $g^{-1}\{0\}$  and has datum  $(-2, 1)$ . Hence the only possible positive real part of a pole of  $Z_\Phi(\omega; f/g)$  is  $\frac{1}{2}$ . Then

Theorem 5.3 predicts the following possible terms in the expansion of  $E_\Phi(z; f/g)$  for sufficiently small  $|z|_K$ :  $\Psi(z)|z|_K^{1/2}$  and  $\Psi(z)|z|_K^{1/2}(-1)^{\text{ord}(z)}$ .

In the concrete case above, one can analogously verify that this is precisely what happens.

**Example 5.15.** We continue similarly the study of case (2) of Example 5.8 when  $K$  is a  $p$ -field. In that case the fibre  $\rho^{-1}\{1\}$  is given by  $x_1 - y_1^2 = 0$  in the chart  $V$  and it has datum  $(1, 1)$ , with respect to the function  $\rho - 1$ . We need to perform two more blow-ups to obtain an embedded resolution of  $\rho^{-1}\{1\} \cup E_1$ . The two extra exceptional curves  $E_2$  and  $E_3$  have data  $(N_2, v_2) = (1, 3)$  and  $(N_2, v_2) = (2, 5)$ , respectively, with respect to the function  $\rho - 1$ .

(1) The fibre  $\rho^{-1}\{0\}$  is again the strict transform of  $f^{-1}\{0\}$  and consists now of one component with datum  $(1, 1)$ . Then again the only possible poles of  $Z_\Phi(\omega; f/g)$  with negative real part must be of the form  $-1 + \frac{2\pi\sqrt{-1}}{\ln q}k, k \in \mathbb{Z}$ , occurring only if  $\chi$  is trivial.

It is well known that  $E_2$  does not contribute to poles of  $Z_\Phi(\omega; f/g - 1)$  (since it intersects only one other component); see for example [14], [35]. Then, by Theorem 3.2 and Remark 3.4, the only possible poles with negative real part of  $Z_\Phi(\omega; f/g - 1)$  are  $-1 + \frac{2\pi\sqrt{-1}}{\ln q}k, k \in \mathbb{Z}$ , occurring only if  $\chi$  is trivial, and  $-\frac{5}{2} + \frac{\pi\sqrt{-1}}{\ln q}k, k \in \mathbb{Z}$ , occurring only if the order of  $\chi$  is 1 or 2. (Clearly all poles are of order 1.)

Hence Theorem 5.12 predicts the following possible terms in the expansion of  $E_\Phi(z; f/g)$  for sufficiently large  $|z|_K$ :  $\Psi(z)|z|_K^{-5/2}$  and  $\Psi(z)|z|_K^{-5/2}(-1)^{\text{ord}(z)}$ . Note again that there should be no terms  $|z|_K^{-1}$  and  $\Psi(z)|z|_K^{-1}$  because of part (2) of that theorem.

As an illustration, we consider now the concrete case  $K = \mathbb{Q}_p$  with  $p \neq 2$ , and  $\Phi$  the characteristic function of  $(p\mathbb{Z}_p)^2$ . The same computation as in (5.4) yields

$$Z_\Phi(\omega; f/g) = \frac{p^{1+s} + p^2(p-2)p^{-s} + p^{2-2s} - 2p + 1}{p^2(p+1)(p^{1+s}-1)(p^{1-2s}-1)}.$$

With a more elaborate computation, using Theorem 3.15, one can verify that  $Z_\Phi(\omega; f/g - 1)$  is

$$\frac{(p-1)(-p^{4+2s} + (p^2 - p + 1)p^{4+s} + (-p^2 + p - 1)p^{2-s} + p^4 - p^3 + p^2 - p + 1)}{p^2(p^{5+2s}-1)(p^{1+s}-1)(p^{1-2s}-1)}.$$

(2) The terms in the expansion of  $E_\Phi(z; f/g)$  for sufficiently small  $|z|_K$  are as in Example 5.14(2).

**5.4. Estimates.** Our main theorems on asymptotic expansions for oscillatory integrals imply estimates for them, for large and small  $|z|_K$ , in terms of the ‘largest negative pole’  $\beta$  and the ‘smallest positive pole’  $\alpha$  of Definition 2.7, respectively. In order to formulate them, we first introduce the orders of these poles.

**Definition 5.16.** We take  $\frac{f}{g} : U \rightarrow K$  and  $\Phi$  as in Section 2.

(1) Take  $\alpha, \beta$  as in Definition 2.7(1), depending on (the support of)  $\Phi$ . When  $\alpha \neq +\infty$ , we put

$$\tilde{m}_\alpha := \max_{\omega} \{\text{order of } \alpha \text{ as pole of } Z_\Phi(\omega; f/g)\},$$



and when  $\beta \neq -\infty$ , we put

$$\tilde{m}_\beta := \max_{\omega} \{\text{order of } \beta \text{ as pole of } Z_\Phi(\omega; f/g)\}.$$

(2) Whenever  $T$  is finite, in particular when  $f$  and  $g$  are polynomials, we can take  $\alpha, \beta$  as in Definition 2.7(2), independent of  $\Phi$ . When  $\alpha \neq +\infty$ , we put

$$\tilde{m}_\alpha := \max_{\omega, \Phi} \{\text{order of } \alpha \text{ as pole of } Z_\Phi(\omega; f/g)\},$$

and when  $\beta \neq -\infty$ , we put

$$\tilde{m}_\beta := \max_{\omega, \Phi} \{\text{order of } \beta \text{ as pole of } Z_\Phi(\omega; f/g)\}.$$

In both cases we call these numbers the *order* of  $\alpha$  and  $\beta$ , respectively. Note that they are less than or equal to  $n$ .

Taking into account Theorem 3.9, the estimates for small and large  $|z|_K$  below follow directly from Theorems 5.2 and 5.3, and Theorems 5.11 and 5.12, respectively.

**Theorem 5.17.** *We take  $\frac{f}{g} : U \rightarrow K$  and  $\Phi$  as in Section 2.*

(1) *Let  $K$  be an  $\mathbb{R}$ -field. We put  $\delta_\alpha = 0$  if  $\alpha \notin \frac{1}{[K:\mathbb{R}]} \mathbb{Z}_{>0}$  and  $\delta_\alpha = 1$  if  $\alpha \in \frac{1}{[K:\mathbb{R}]} \mathbb{Z}_{>0}$ .*

*If  $Z_\Phi(\omega; f/g)$  has a positive pole for some  $\omega$ , then there exists a positive constant  $C$  such that*

$$\left| E_\Phi(z; f/g) - \int_{K^n} \Phi(x) |dx|_K \right| \leq C |z|_K^\alpha |\ln |z|_K|^{\tilde{m}_\alpha + \delta_\alpha - 1} \text{ as } |z|_K \rightarrow 0.$$

(2) *Let  $K$  be  $p$ -field. If  $Z_\Phi(\omega; f/g)$  has a pole with positive real part for some  $\omega$ , then there exists a positive constant  $C$  such that*

$$\left| E_\Phi(z; f/g) - \int_{K^n} \Phi(x) |dx|_K \right| \leq C |z|_K^\alpha |\ln |z|_K|^{\tilde{m}_\alpha - 1} \text{ for sufficiently small } |z|_K.$$

Here in general  $\alpha$  and  $\tilde{m}_\alpha$  must be interpreted in the sense of Definitions 2.7(1) and 5.16(1), that is, depending on  $\Phi$ . Then also  $\delta_\alpha$  and  $C$  depend on  $\Phi$ . Whenever  $T$  is finite, in particular when  $f$  and  $g$  are polynomials, one can consider  $\alpha$  and  $\tilde{m}_\alpha$  in the sense of Definitions 2.7(2) and 5.16(2). Then  $C$ ,  $\alpha$ ,  $\delta_\alpha$  and  $\tilde{m}_\alpha$  are independent of  $\Phi$ .

A similar remark applies to the next theorem.

**Theorem 5.18.** *We take  $\frac{f}{g} : U \rightarrow K$  and  $\Phi$  as in Section 2. Denote for  $c \in \mathcal{V}$  the corresponding number  $\beta$ , associated in Definition 2.7 to  $f/g - c$ , by  $\beta_c$ . We define  $\beta_{\mathcal{V}} := \max_{c \in \mathcal{V}} \beta_c$  and its associated order  $\tilde{m}_{\beta_{\mathcal{V}}}$  as in Definition 5.16.*

*Assume that for some  $\omega$  and some  $c \in \mathcal{V}$ , the zeta function  $Z_\Phi(\omega; f/g - c)$  has a negative pole or a pole with negative real part, according as  $K$  is an  $\mathbb{R}$ -field or a  $p$ -field, for some  $\omega$ . Then there exists a positive constant  $C$  such that*

$$|E_\Phi(z; f/g)| \leq C |z|_K^{\beta_{\mathcal{V}}} |\ln |z|_K|^{\tilde{m}_{\beta_{\mathcal{V}}} - 1} \text{ as } |z|_K \rightarrow \infty.$$

**Remark 5.19.** Let  $K$  be a  $p$ -field. In the classical case, i.e., with  $g = 1$  and with  $\Phi$  the characteristic function of  $R_K^n$ , the oscillatory integral  $E_\Phi(z; f/g)$  becomes a traditional exponential sum. This fact is not true for a general  $g$ . Consider however the particular case of a non-degenerate Laurent polynomial  $h(x) = \frac{f(x)}{x^m}$ ,

where  $f \in R_K[x_1, \dots, x_n] \setminus R_K$  and  $x^m = \prod_{i=1}^n x_i^{m_i}$  (all  $m_i \in \mathbb{Z}_{\geq 0}$ ), and the associated exponential sum

$$S_{\ell,u}(h) := q^{-\ell n} \sum_{\bar{x} \in ((R_K/P_K^\ell)^\times)^n} \Psi(u\mathfrak{p}^{-\ell}h(\bar{x})),$$

where  $\ell \in \mathbb{Z}_{>0}$  and  $u \in R_K^\times$ . Then we have

$$(5.5) \quad S_{\ell,u}(h) = \int_{(R_K^\times)^n} \Psi(u\mathfrak{p}^{-\ell}h(x)) |dx|_K.$$

Indeed, decompose  $(R_K^\times)^n$  as

$$(R_K^\times)^n = \coprod_{\bar{x} \in ((R_K/P_K^\ell)^\times)^n} \tilde{x} + \mathfrak{p}^\ell R_K^n,$$

where  $\tilde{x}$  is a representative of  $\bar{x}$ . Then on each piece of the decomposition we have for all  $y \in R_K^n$  that  $h(\tilde{x} + \mathfrak{p}^\ell y)$  is of the form

$$\frac{f(\tilde{x} + \mathfrak{p}^\ell y)}{(\tilde{x} + \mathfrak{p}^\ell y)^m} = \frac{f(\tilde{x}) + \mathfrak{p}^\ell A}{\tilde{x}^m + \mathfrak{p}^\ell B} = \frac{f(\tilde{x}) + \mathfrak{p}^\ell A}{\tilde{x}^m(1 + \mathfrak{p}^\ell B')} = \frac{f(\tilde{x}) + \mathfrak{p}^\ell C}{\tilde{x}^m} = \frac{f(\tilde{x})}{\tilde{x}^m} + \mathfrak{p}^\ell D,$$

where  $A, B, B', C, D \in R_K$ , and the second and last equalities are valid because the components of  $\tilde{x}$  are units. Hence  $h(\tilde{x} + \mathfrak{p}^\ell y)$  is of the form  $h(\tilde{x}) + \mathfrak{p}^\ell D$ . This implies (5.5).

We can apply Theorem 5.18 to the meromorphic function  $h$  on  $R_K^n$  and the characteristic function of  $(R_K^\times)^n$  to obtain the estimate

$$(5.6) \quad |S_{\ell,u}(h)| \leq C q^{\ell\beta_V} \ell^{\tilde{m}_{\beta_V}-1} \quad \text{for sufficiently large } \ell,$$

where  $C$  is a positive constant, and  $\beta_V$  and  $m_{\beta_V}$  are as in Theorem 5.18. Note however that this result can already be obtained using Igusa's classical method for estimating exponential sums mod  $p^\ell$ , due to the fact that  $h$  is a regular function on  $(R_K^\times)^n$ .

The estimation (5.6) can be considered as a  $p$ -adic (or mod  $p^\ell$ ) counterpart of the estimations for exponential sums attached to Laurent polynomials over finite fields, due to Denef and Loeser [17].

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